Ecka punt voor gemaakte opg:

10% → ecka point

20% → 2p

40% → 3p

60% → 4p

80% → 5p

90% → 6p

Exam: 4 questions, each max 6 points

Defin

1 Examples and basic concepts

1.1 Dynamical systems

Def A pair (X, T) where X is a set and $T: X \to X$ is a map from X to itself is called a discrete (time) dynamical system. The set X is sometimes called the phase space and the mapping is the associated dynamics

for the n^{th} iterate of the mapping T we use the notation

$$T^n = \underbrace{T \circ \dots \circ T}_{n \text{ times}}$$

for any $n \in \mathbb{N}$, with the convention that $T^0 = id$. If the mapping T is invertible, we use the notation

$$T^{-n} = \underbrace{T^{-1} \circ \dots \circ T^{-1}}_{n \text{ times}}$$

for the n^{th} iterate of the inverse of T.

1.1.2 Definition A pair (X, T^t) where X is a set and $T^t: X \to X$, $t \in \mathbb{R}$ is a 1-parameter family of maps, which form a group, that is $T^0 = id$ and for any $s, t \in \mathbb{R}$ we have $T^{s+t} = T^s \circ T^t$, is a continious (time) dynamical system or a flow.

We use the same definition also in the case where $t \in \mathbb{R}^+_0$. The only difference is that the 1-parameter family of maps T^t technically forms a semigroup instead of a group. In the case where $t \in \mathbb{R}^+_0$, the pair (X, T^t) is called

a semiflow.

1.1.3 Remarks

a) Often the space X have some extra structure and this structure is preserved by the transformation T. For example
   - X is a topological space and T is continious,
   - There is a measure μ on the space X, which is invariant under T, meaning that for any measurable $A \subset X$ we have $\mu(T^{-1}(A)) = \mu(A)$
   - X is a differentiable manifold and T is differentiable

b) for a fixed t_0, the pair (X, T^{t_0}) is a discrete time dynamical system.

c) In these notes, the generic term dynamical system means a pair (X, T^t) where $t \in \mathbb{N}, \mathbb{Z}, \mathbb{R}$ or $\mathbb{R}_0^+$. The cases $t \in \mathbb{N}$ and $t \in \mathbb{Z}$ correspond to discrete time dynamical systems and the remaining two to continious time dynamical systems.

1.1.4 Definition Let (X, T^t) be a dynamical system. The positive orbit of a point $x \in X$ is the set

$$\mathcal{O}_T^+(x) = \bigcup_{t \geq 0} \{T^t(x)\}.$$

If the mapping T is invertible, then the negative orbit of a point $x \in X$ is the set

$$\mathcal{O}_T^-(x) = \bigcup_{t \leq 0} \{T^t(x)\}$$

and the orbit of x is

$$\mathcal{O}_T(x) = \mathcal{O}_T^+(x) \cup \mathcal{O}_T^-(x)$$

1.1.5 Remark If T is not invertible, then we may use the term orbit from the positive orbit of T, but the meaning should be clear from context

1.1.6 Definition Let $t > 0$ and $x \in X$. The point x is

1) t-periodic if $T^t(x) = x$,

2) a fixed point (of T) if $T^s(x) = x$, for any s.

The minimal period of a periodic point x is the minimal $t > 0$, s.t. $T^t(x) = x$.

Notice that if a point $x \in X$ is t-periodic, then it is nt-periodic for any positive integer n, and that a fixed point x is t-periodic $\forall t > 0$.

1.1.7 Definition A subset $A \subset X$ is (T-) invariant if $T^t(A) \subset A$, $\forall t$. Similarly, A is called forward invariant if $T^t(A) \subset A$, $\forall t \geq 0$ and Backward invariant if $T^{-t}(A) \subset A$, $\forall t \geq 0$. Here we use the convention $T^{-t}(A) = (T^t)^{-1}(A)$ so the last definition still makes sense even if T is not invertible.

1.1.8 Example

a) Let X be a set and $T = id$. Then (X, T) is a "static system"

In particular $T(x) = x \ \forall x \in X$, so every point is a fixed point, and thus n-periodic $\forall n \in \mathbb{N}$. Also for any $A \subset X$ we have $T(A) = A$, so every set is invariant.

b) Let $S^1 = [0,1]/_\sim$, where $\sim$ is the equivalence relation defined by $0 \sim 1$. Then $d(x,y) = \min\{|x-y|, 1-|x-y|\}$ is a natural metric on S^1 and $\lambda|_{[0,1]} = \lambda$ is a natural measure on S^1. Let $\alpha \in \mathbb{R}$ and define $R_\alpha : S^1 \to S^1$ by $R_\alpha(x) = (x+\alpha) \bmod 1$. The map R_α is the rotation by the angle α. Note that $d(R_\alpha(x), R_\alpha(y)) = d(x,y) \ \forall x,y \in S^1$ and $(R_\alpha)_* \lambda = \lambda$? Thus the map R_α is an isometry which preserves the Lebesgue measure λ. If $\alpha \in \mathbb{Q}$, then $\alpha = \frac{p}{q}$. Thus $R_\alpha^q = id$ and (no logisch) every point is q-periodic. If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then every forward (or backward) orbit is dense in S^1, that is $\overline{O^+_{R_\alpha}(x)} = S^1$ $\forall x \in S^1$.

1.2 Conjugacy and semiconjugacy

1.2.1 Definition Two dynamical systems (Y, S) and (X, T) are conjugate or isomorphic, if there exists a bijection $\pi : Y \to X$, s.t. $T^t \circ \pi = \pi \circ S^t \ \forall t$. Then π is called a conjugacy or an isomorphism, and we say that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ \downarrow \pi & \circlearrowleft & \downarrow \pi \\ X & \xrightarrow{T} & X \end{array}$$

commutes. If π is only a surjection, then it's called

semiconjugacy. In this case (X,T) is a factor of (Y,S) and (Y,S) is an extension of (X,T)

1.2.2. Example Let $b \in \mathbb{Z} \setminus \{0\}$. Then there exists a continuous semiconjugacy π from (S^1, R_α) to $(S^1, R_{b\alpha})$.

proof: Define $\pi: S^1 \to S^1$ by setting $\pi(x) = bx \bmod 1$.
Then π is clearly a continuous surjection. Furthermore, $\forall x \in S^1$,

$$\pi \circ R_\alpha(x) = \pi((x+\alpha) \bmod 1) = (b((x+\alpha) \bmod 1)) \bmod 1$$

$$= (b(x+\alpha)) \bmod 1 = (bx + b\alpha) \bmod 1$$

$$= bx \bmod 1 + b\alpha = R_{b\alpha}(bx \bmod 1)$$

$$= R_{b\alpha} \circ \pi(x)$$

QED

1.2.3 Definition Let (X,d) be a metric space. A map $T: X \to X$ is (uniformly) expanding, if $\exists \mu > 1$, $\delta > 0$ s.t.

$$d(T(x), T(y)) \geq \mu\, d(x,y)$$

$\forall x, y \in X$ with $d(x,y) < \delta$.

1.2.4 Example Let $b \in \mathbb{Z}$ be s.t. $|b| > 1$. Define the map $E_b: S^1 \to S^1$ by setting $E_b(x) = bx \bmod 1$. Then E_b is expanding.

proof: Let $\delta = \frac{1}{2b}$, $\mu = b$ and Let $x, y \in S^1$ be s.t. $d(x,y) < \delta$. Let $\alpha \in \mathbb{R}$ be s.t. $R_\alpha(x) = \frac{1}{2}$. Then (recall that R_α is an isometry) we have that $|bR_\alpha(x) - bR_\alpha(y)| < \frac{1}{2}$. Because $d(R_\alpha(x), R_\alpha(y)) \overset{\text{isom.}}{=} d(x,y) < \delta = \frac{1}{2b}$ and 1.1.8(ii) (wlog?). In Example 1.2.2 we saw that $E_b \circ R_\alpha \overset{\text{s}}{=} R_{b\alpha} \circ E_b$. Since $R_{b\alpha}$ is also an isometry, we obtain that.

$$
\begin{aligned}
d(E_b(x), E_b(y)) &\overset{\text{isom.}}{=} d(R_{b\alpha}(E_b(0)), R_{b\alpha}(E_b(y))) \\
&\overset{\text{s}}{=} d(E_b(R_\alpha(x)), E_b(R_\alpha(y))) \\
&= |bR_\alpha(x), bR_\alpha(y)| \\
&= b\, d(R_\alpha(x), R_\alpha(y)) \overset{\text{isom.}}{=} b\, d(x,y) = \mu d(x,y).
\end{aligned}
$$

QED

Observe that $(E_b)_\# d = d(E)$. Note also that $d(E_b(I)) = b\,d(I)$ if I is a short interval. Further, 0 is a fixed points and $E_b^n\left(\frac{r}{b^n}\right) = 0$.

1.3. Symbolic space and the examples

1.3.1 Definition Let $m \in \mathbb{N}$, $m > 1$ and set $I = \{0, 1, \dots, m-1\}$. A sequence space or symbolic space in $\Sigma_m = I^{\mathbb{Z}}$ or $\Sigma_m^+ = I^{\mathbb{N}}$. The elements of I are called symbols and for $b \in \mathbb{N}$, the elements of I^b are called words. If $w \in I^b$, the length of w is $|w| = b$. The (left) shift is

a map $\sigma: \Sigma_m \to \Sigma_m$ or $\sigma: \Sigma_m^+ \to \Sigma_m^+$ defined by $\sigma(x)_i = x_{i+1}$.

1.3.2 Example In Σ_m the left shift σ is a bijection, but on Σ_m^+ it is only a surjection and every point has m preimages. (σ (in Σ_m^+) is not inj count $\sigma(1,2,3,4,\dots) = 2,3,4 = \sigma(2,2,3,4\dots)$ $\underbrace{\text{so even } m \text{ preimages}}$.

If we equip Σ_m (or Σ_m^+) with the product topology (on I we have the discrete topology), then Σ_m is a compact topological space.

The basis is given by cylinder sets of the form

$$C_\omega = \{x \in \Sigma_m \mid x_i = \omega_i \ \forall i \in \{-\ell, \dots, n\}\},$$

where $\omega \in \prod_{i=-\ell}^{n} I$. A metric $d(x,y) = m^{-\min\{|i| \mid x_i \neq y_i\}}$ induces the topology.

In this topology, the shift map σ is continuous on Σ_m^+ (E) and a homeomorphism on Σ_m. It is also easy to see that on Σ_m^+, the map σ is expanding. Let us now show that the periodic points of σ are dense.

proof: It is enough to show that for all $\varepsilon > 0$ and $x \in \Sigma_m$, there exists a periodic point $y \in \Sigma_m$, such that $d(x,y) < \varepsilon$.

Fix $x \in \Sigma_m$ and choose $\ell \in \mathbb{N}$ large enough, s.t. $m^{-\ell} < \varepsilon$.

Define the word y by setting $y = \dots www\dots$, or more explicitly

$y_i = x_{i+\ell \pmod{2\ell+1}-\ell}$. Then we have $d(x,y) \leq m^{-\ell} < \varepsilon$ and clearly $\sigma^{2\ell+1}(y) = y$, so y is $(2\ell+1)$-periodic which finishes the proof

QED ∎

1.3.3 proposition There is a continuous semiconjugacy between (Σ_m^+, σ) and (S^1, E_m).

proof: Define $\pi : \Sigma_m^+ \to S^1$ by $\pi(x) = \sum_{i=0}^{\infty} \frac{x_i}{m^{i+1}}$. $(\in [0,1])$

clearly π is continuous and surjective and it is injective except that

$$\pi(\omega_1, \dots, \omega_b, m-1, m-1, \dots) = \pi(\omega_1, \dots, \omega_{b-1}, \omega_b + 1, 0, 0, \dots).$$

want:

$$LL = \frac{\omega_1}{m} + \dots + \frac{\omega_b}{m^b} + \frac{m-1}{m^{b+1}} + \frac{m-1}{m^{b+2}} + \dots$$

$$RL = \frac{\omega_1}{m} + \dots + \frac{\omega_{b-1}}{m^{b-1}} + \frac{\omega_b + 1}{m^b} \qquad \text{in} \quad \sum_{i=1}^{\infty} \frac{m-1}{m^{b+i}} = \frac{1}{m^b}$$

Finally, for any $x \in \Sigma_m^+$ we have

$$E_m \circ \pi(x) = \left(m \sum_{i=0}^{\infty} \frac{x_i}{m^{i+1}}\right) \bmod 1 \overset{\frac{m}{m^{i+1}} = \frac{1}{m^i}}{=} \left(x_0 + \sum_{i=1}^{\infty} \frac{x_i}{m^i}\right) \bmod 1$$

$$= \sum_{i=0}^{\infty} \frac{x_{i+1}}{m^{i+1}} = \sum_{i=0}^{\infty} \frac{\sigma(x)_i}{m^{i+1}} = \pi \circ \sigma(x),$$

which shows that π is a semiconjugacy. QED ∎

1.3.4 Remark By proposition 1.3.3 it is easy to see that the system (S^1, E_m) has b-periodic points for any $b \in \mathbb{N}$, periodic points are dense in S^1, there are dense orbits (↦ D1.3.2), etc.

Definition 1.3.5 Let X be a topological space and $T: X \to X$ a continuous map. A fixed point $p \in X$ is called attractive if there is an open neighbourhood U of p s.t. $\overline{U}$ is sequentially compact, $T(\overline{U}) \subset U$ and

$$\bigcap_{n \geq 0} T^n(U) = \{p\}.$$

A fixed point $q \in X$ is called repelling if there exists an open neighbourhood U of q, s.t. $\overline{U}$ is sequentially compact, $\overline{U} \subset T(U)$ (expanding) and

$$\bigcap_{n \geq 0} \overline{T}^n(U) = \{q\}.$$

A fixed point $p \in X$ is isolated if there is an open neighbourhood U of p, s.t. p is the only fixed point in U.

note: An isolated point can be isolated and repelling or attractive.

1.3.6 Example

a) Let $T: [-1,1] \to [-1,1]$, defined by $T(x) = \frac{x}{2}$. Clearly 0 is a fixed point of T. Furthermore, since $]-1,1[$ is an open set containing 0, with $\overline{]-1,1[} = [-1,1]$ being sequentially compact as a closed and bounded subset of $\mathbb{R}$,

$T([-1,1]) = [-\frac{1}{2}, \frac{1}{2}] \subset]-1,1[$ and

$$\bigcap_{n\geq 0} T^n(]-1,1[) = \{0\}, \quad \text{(easy to see)}$$

we see that 0 is an attractive fixed point. It is also easy to see that 0 is the only fixed point in $]-1,1[$, so in particular it is isolated.

b) The family of maps $T_\mu : \mathbb{R} \to \mathbb{R}$, $\mu > 0$ defined by

$$T_\mu(x) = \mu x(1-x) = \mu x - \mu x^2,$$

is called a logistic family or quadratic family. It contains a lot of different kinds of dynamics for different values of the parameter μ. Observe that if $\mu > 1$ and $x \in \mathbb{R} \setminus [0,1]$, then

$$\lim_{n\to\infty} T_\mu^n(x) = -\infty.$$

If $\mu \in [0,4]$ then $T_\mu([0,1]) \subset [0,1]$ (Desmos). If $\mu \in [0,1]$, then

$$|T_\mu'(x)| = |\mu - 2\mu x| < \mu \leq 1,$$

$\forall x \in]0,1[$. This implies that $d(T_\mu(x), 0) < \mu\, d(x,0)$ (?) and thus $T_\mu^n(x) \xrightarrow{n\to\infty} 0$, $\forall x \in [0,1]$ and $\bigcap_{n\geq 0} T_\mu^n([0,1]) = \{0\}$. In other words, 0 is an attractive fixed point. If $\mu > 1$, then T_μ has two fixed points: 0 and $1 - \frac{1}{\mu}$. Since $T_\mu'(0) = \mu$ and $T_\mu'(1-\frac{1}{\mu}) = 2-\mu$,

The point 0 is a repelling fixed point and $1-\frac{1}{\mu}$ is an attractive fixed point provided that $\mu \in]1,3[$. If $\mu > 4$, then T_μ has a quite complicated behaviour.

1.3.7 Example Let $A: \mathbb{R}^2 \to \mathbb{R}^2$ be the map $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, that is $A(x,y) = A\begin{pmatrix} x \\ y \end{pmatrix}$. The eigenvalues of A are $\lambda = \frac{3+\sqrt{5}}{2} > 1$ and $\frac{1}{\lambda} < 1$. The corresponding eigenvectors are $v_\lambda = \left(\frac{1+\sqrt{5}}{2}, 1\right)$ and $v_{1/\lambda} = \left(\frac{1-\sqrt{5}}{2}, 1\right)$. Since $A(\mathbb{Z}^2) \subset \mathbb{Z}^2$, the matrix A induces a map on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ via the formula

$$A(x,y) = ((2x+y) \bmod 1, (x+y) \bmod 1).$$

Since $\det A = 1$, the inverse A^{-1} is an integer matrix and thus defines a map $A^{-1}: \mathbb{T}^2 \to \mathbb{T}^2$ and A is a bijection.

Let $W^u(x) = x + \langle v_\lambda \rangle$ and $W^s(x) = x + \langle v_{1/\lambda} \rangle$. Then

$$A(W^u(x)) = Ax + \langle v_\lambda \rangle = W^u(Ax)$$

and $A(W^s(x)) = W^s(Ax)$. The set $W^u(x)$ is called the unstable manifold of A at x and $W^s(x)$ is the stable manifold of A at x. Since the tangent of $W^u(x)$

is irrational, $W^u(x)$ is dense on T^2 and similarly for $W^s(x)$.
These manifolds $W^u(x)$, $x \in T^2$ (or $W^s(x)$, $x \in T^2$) form a partition of T^2 and it is called an unstable (stable) foliation of T^2. This is classical example of a hyperbolic dynamical system.

1.3.8 Definition Let X be a topological space and let $T: X \to X$ be a continuous map. A compact set $C \subset X$ is an attractor of T, if there exists an open set $U \supset C$ s.t. $\bar{U}$ is sequentially compact, $T(\bar{U}) \subset U$ and

$$C = \bigcap_{n \geq 0} T^n(U).$$

The basin of attraction of C is

$$B(C) = \bigcup_{n \geq 0} T^{-n}(U).$$

A trapping region is an open set V s.t. $\bar{V}$ is sequentially compact and $T(\bar{V}) \subset V$.

1.3.9 Remark If C is an attractor, then $T(C) = C$.

Proof: $\subseteq$) $T(C) = T\left(\bigcap_{n \geq 0} T^n(U)\right) \subset \bigcap_{n \geq 1} T^n(U) \overset{T(U) \subset U}{=} \bigcap_{n \geq 0} T^n(U) = C$.

$\supseteq$) Let $x \in C$. Since $T(\bar{U}) \subset U$, we have that

$T^n(\bar{U}) \subset U \subset \bar{U}$ $\forall n \in \mathbb{N}$. Since $x \in T^n(U)$ $\forall n$,

$\exists y_n \in T^{n-1}(U) \subset T^{n-1}(\bar{U}) \subset \bar{U}$ s.t. $T(y_n) = x$. Let

(y_{n_j}) be a convergent subsequence of (y_n) and let

$y = \lim_{j \to \infty} y_{n_j}$. Since T is continuous, we have that

$$T(y) = \lim_{j \to \infty} T(y_{n_j}) = x.$$

Since $T^b(\bar{U})$ is sequentially compact for every b and

$y_{n_j} \in T^b(\bar{U})$ for all large j, we have that

$y \in T^b(\bar{U}) \subset T^{b-1}(U)$ $\forall b$. Therefore $y \in C$.

1.3.10 Example

a) Let $T: [-1,1] \to [-1,1]$, $T(x) = \frac{x}{2}$. Then $\{0\}$ is an attractor since $\bigcap_{n \geq 0} T^n(]-1,1[) = \{0\}$

b) Let $T: [-2,2] \to [-2,2]$, $T(x) = \frac{1}{3}x^3 - \frac{4}{3}x$

$= \frac{1}{3}(x-2)(x+2)$. Let $U =]-\frac{5}{4}, -\frac{3}{4}[\cup]\frac{3}{4}, \frac{5}{4}[$. Note that

$T(\bar{U}) \subset U$, since $|T(\pm\frac{3}{4})| > \frac{3}{4}$ and $|T(\pm\frac{5}{4})| < \frac{5}{4}$. It

is easy to see that

$$\bigcap_{n \geq 0} T^n(U) = \{-1, 1\},$$

hence $\{-1, 1\}$ is an attractor.

c) Let $T = S^1 \times D^2$ be a solid torus where

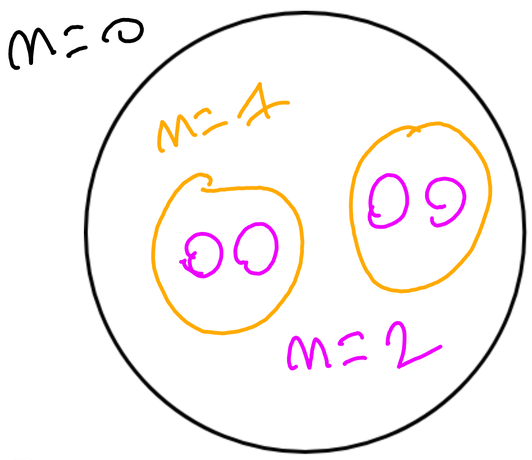
→ doughnut

$D^2 = \{z \in \mathbb{R}^2 \mid |z| \leq 1\}$. Let $d \in]0, \frac{1}{2}[$ and define

$T : \mathcal{T} \to \mathcal{T}$ by setting

$$T(\varphi, x, y) = (2\varphi \bmod 1, dx + \tfrac{1}{2}\cos(2\pi\varphi), dy + \tfrac{1}{2}\sin(2\pi\varphi)).$$

The map T expands by the factor 2 in the direction S^1 and contracts by the factor d in the direction D^2. Further, T is $d \in]0, \frac{1}{2}[$ an injection and $T(\mathcal{T}) \subset \operatorname{Int} \mathcal{T}$. Note that a cross section of $T^n(\mathcal{T})$ consists of 2^n disjoint discs with radius d^n.



The attractor $S = \bigcap_{n \geq 0} T^n(\mathcal{T})$ exists, and is compact since each $T^n(\mathcal{T})$ is compact. The set S is called a solenoid. The map $T|_S$ is a bijection and locally S is homeomorphic to $I \times C$, where C is a cantor set.

1.3.11 definition Let $T : X \to X$ be a map and $c : X \to \mathbb{R}^+$ a positive map. Define

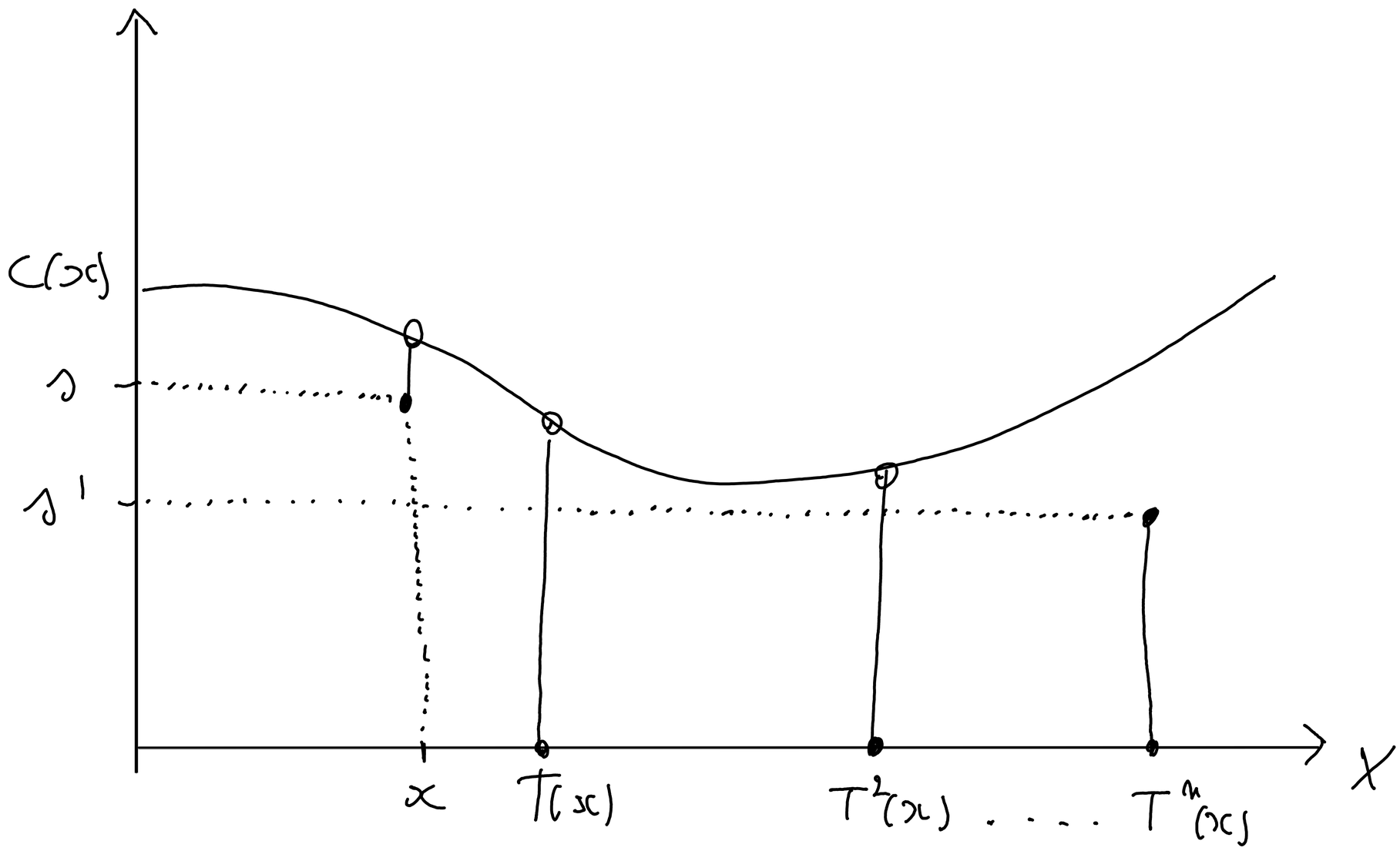
$$\tilde{X}_c := \{(x, t) \in X \times \mathbb{R}^+ \mid 0 \leq t \leq c(x)\}/\sim$$

where the equivalence relation $\sim$ is given by $(x, c(x)) \sim (T(x), 0)$ $\forall x \in X$. A Suspension of the map T is determined by the ceiling function c is the semiflow $\phi^t : \tilde{X}_c \to \tilde{X}_c$ defined by

$$\phi^t(x, s) = (T^n(x), s'),$$

where $n \in \mathbb{N}$ and $s' \in \mathbb{R}$ are the unique numbers satisfying

$$\sum_{i=0}^{n-1} c(T^i(x)) + s' = t + s, \quad 0 \leq s' < c(T^n(x)).$$



1.3.12 Definition A cross section of a (semi)flow $\phi^t : Y \to Y$ is a subset $X \subset Y$ s.t. the set

$$A_y := \{t \in \mathbb{R}^+ \mid \phi^t(y) \in X\},$$

is non-empty and discrete $\forall y \in Y$. For all $x \in X$

define $\tau(x) = \min A_x$ which is called the (first) return time of x. The (first) return map or the Poincaré (return) map is the map $T: X \to X$ defined by $T(x) = \phi^{\tau(x)}(x)$.

1.3.13 Remark The suspension and the Poincaré map are inverse constructions of each other.

1.3.14 Example Let $T^2 = S^1 \times S^1$ be the torus, $\alpha \in \mathbb{R}$ and define $\phi_\alpha^t : T^2 \to T^2$ by

$$\phi_\alpha^t(x,y) = (x + \alpha t, y + t) \bmod 1.$$

Then ϕ_α^t is conjugate to the suspension of R_α in Example 1.1.8, with the ceiling function $c(x) = 1$, for all $x \in T^2$. Also $S^1 \times \{0\}$ is a cross section of ϕ_α^t with a return time $\tau(x) = 1 \ \forall x \in S^1 \times \{0\}$ and the corresponding Poincaré map is R_α.

2 Topological Dynamics

2.0.1 Definition A topological dynamical system is a pair (X, T) where X is a topological space and

$T: X \to X$ is a continuous map. In the case of (semi)flows we assume that the map

$$(x,t) \to T^t(x)$$

is continuous. A topological (semi) conjugacy is a (semi) conjugacy which is (continuous) homeomorphism.

From this point on, we assume that X is a separable locally compact metric space with the topology induced by the metric, and that T is continuous.

2.1 Limit sets and recurrence

2.1.1 Definition Let $x \in X$. A point $y \in X$ is a ω-limit point of x if there exists a sequence of integers $n_k \xrightarrow{k\to\infty} \infty$ s.t.

$$T^{n_k}(x) \xrightarrow{k\to\infty} y.$$

The ω-limit set of x is the collection of the ω-limit points and is denoted by $\omega(x)$. That is

$$\omega(x) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i\geq n} \{T^i(x)\}}.$$

If the map T is invertible, then the α-limit set of x

is

$$\alpha(x) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i \geq n} \{T^{-i}(x)\}},$$

and its elements are called the α-limit points of x. A point $x \in X$ is called recurrent if $x \in \omega(x)$. The set of recurrent points is denoted by $R(T)$. A point x is called non-wandering if for all open sets $U \ni x$, there is an integer $n \in \mathbb{N} \setminus \{0\}$ s.t.

$$T^n(U) \cap U \neq \emptyset.$$

The set of non-wandering points is denoted by $NW(T)$.

2.1.2 Theorem Let (X,T) be a topological dynamical system. Then the set $R(T)$ is forward invariant, the set $NW(T)$ is closed and forward invariant, and we have

$$\overline{R(T)} \subset NW(T).$$

Furthermore, $\omega(x), \alpha(x) \subset NW(T)$ $\forall x \in X$.

Proof: We begin by showing that $T(R(T)) \subset R(T)$. (forward inv.)
Let $x \in R(T)$, so by definition there is a sequence of integers $n_k \to \infty$, s.t. $T^{n_k}(x) \to x$.

Since T is continuous, we have

$$T(x) = T\left(\lim_{k\to\infty} T^{n_k}(x)\right) = \lim_{k\to\infty} T\left(T^{n_k}(x)\right) = \lim_{k\to\infty} T^{n_k}(T(x)),$$

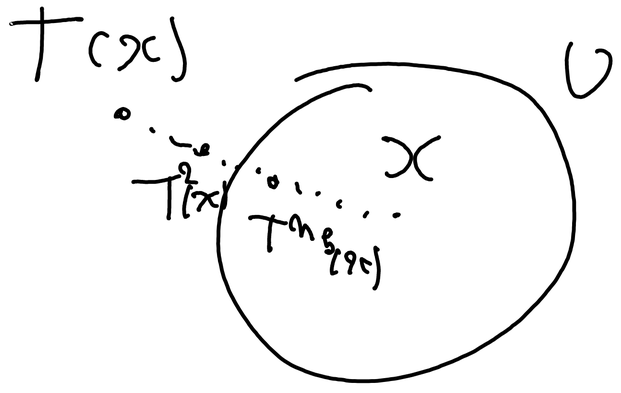
which shows that $T(x) \in R(T)$ proving the claim. (R(T) is fwinv)

Observe now that if $x \notin NW(T)$, then there exists an open set $U \ni x$ s.t. $T^n(U) \cap U = \emptyset$ $\forall n \in \mathbb{N}\setminus\{0\}$. Now any $y \in U$ satisfies $y \notin NW(T)$. Hence $NW(T)^C$ is open which means that $NW(T)$ is closed. The set $NW(T)$ is also forward invariant (E). Next we will show that $\overline{R(T)} \subset NW(T)$

To this end let $x \in R(T)$. Then there exists a sequence $n_k \xrightarrow{k\to\infty} \infty$ s.t.

$$T^{n_k}(x) \xrightarrow{k\to\infty} x.$$

Let $U \ni x$ be an open set. Then $T^{n_k}(x) \in U$ for all large enough k, so $T^{n_k}(U) \cap U \neq \emptyset$.



Thus $x \in NW(T)$. Since $NW(T)$ is closed, $\overline{R(T)} \subset NW(T)$. Proofs of the inclusions $\omega(x), \alpha(x) \subset NW(T)$ are left as an exercise.

QED

2.1.3 Proposition

a) If $T: X \to X$ is a homeomorphism, $y \in \overline{\mathcal{O}(x)}$ and $z \in \overline{\mathcal{O}(y)}$, then $z \in \overline{\mathcal{O}(x)}$

b) If $T: X \to X$ is continuous, $y \in \overline{\mathcal{O}^+(x)}$ and $z \in \overline{\mathcal{O}^+(y)}$, then $z \in \overline{\mathcal{O}^+(x)}$

proof: a) (E)

b) Let $U \ni z$ be open. Since $z \in \mathcal{O}^+(y)$, then exists $n \in \mathbb{N}$ s.t. $\overset{\text{by orbit}}{\overset{\parallel}{z}} = T^n(y) \in U$, so that $y \in T^{-n}(U)$. Since $T^{-n}(U)$ is open and $y \in \overline{\mathcal{O}^+(x)}$, there exists $m \in \mathbb{N}$ s.t. $\overset{\text{by}}{\overset{\parallel}{T^m}}(x) \in T^{-n}(U)$. Therefore $T^{n+m}(x) \in U$. wrm $z \in \overline{\mathcal{O}^+(x)}$?

QED ∎

2.1.4 Definition

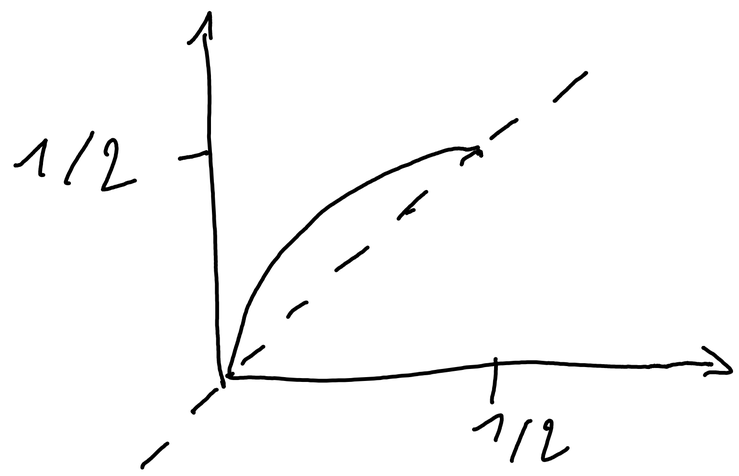
A closed, non-empty and forward invariant set $\gamma \subset X$ is minimal, if it does not contain any closed non-empty forward invariant set as a proper subset. If X is minimal, then we say that T is minimal.

2.1.5 Example

a) Let $E_m: S^1 \to S^1$, $E_m(x) = mx \bmod 1$. Then $E_m(\{0\}) = \{0\}$, so $\{0\}$ is a minimal set.

b.) Let $X =]0, \frac{1}{2}[$ and let $T: X \to X$ be the map

$T(x) = 2x(1-x)$. By problem 7 of exercise 2, $T(x) > x \ \forall x \in X$.



If $Y \subset X$ is a closed forward invariant set and $x \in Y$, then $T(x) \in Y$ and the set $Y \cap [T(x), \frac{1}{2}]$ is closed, non-empty and forward invariant.

2.1.6 Theorem Let $X \neq \emptyset$ be compact and let $T: X \to X$ be continuous. Then there exists a minimal set.

proof: Let $\mathcal{C}$ be the collection of closed non-empty forward invariant subsets of X equipped with the order induced by set inclusion. Clearly, $X \in \mathcal{C}$ so $\mathcal{C} \neq \emptyset$. Let $\mathcal{K} \subset \mathcal{C}$ be totally ordered. Then every finite intersection of elements of $\mathcal{K}$ is non-empty. Since X is compact, we have that

$$\bigcap_{K \in \mathcal{K}} K \neq \emptyset.$$

This intersection is a lower bound of $\mathcal{K}$. By Zorn's lemma there is a minimal el$^{\underline{t}}$ in C. This minimal el$^{\underline{t}}$ is a minimal set.

QED ∎

2.1.7 Remark A compact forward invariant set Y is minimal, iff the orbit of every point in Y is dense in Y. (E) Therefore if T has a minimal set, then $R(T) \neq \emptyset$.

2.1.8 Definition A set $A \subset \mathbb{N}$ is relatively dense or syndetic, if $\exists\, B \in \mathbb{N} \setminus \{0\}$ s.t.

$$\{n, n+1, \dots, n+B\} \cap A \neq \emptyset$$

for every $n \in \mathbb{N}$. A point $x \in X$ is almost periodic, if for all open sets $U \ni x$ the set

$$\{i \in \mathbb{N} \mid T^i(x) \in U\}$$

is relatively dense.

$\rightarrow \{i \mid T^i(x) = x \overset{!}{\in} U\} = \mathbb{N}$

2.1.9 Remark A periodic point is almost periodic, but there are almost periodic points which are not periodic.

2.1.10 Theorem Let X be compact and $T: X \to X$ be continuous. Then $\overline{\mathcal{O}^+(x)}$ is minimal $\iff$ x is almost periodic.

Proof: First we observe that since T is continuous, we have that

$$T(\overline{\mathcal{O}^+(x)}) \subset \overline{\mathcal{O}^+(x)}.$$

Thus $\overline{\mathcal{O}^+(x)}$ is a non-empty, closed and forward invariant set. It is also compact as a closed subset of a compact set.

$\Longleftarrow$ Let x be almost periodic and $y \in \overline{\mathcal{O}^+(x)}$. By Remark 2.1.7 (phrase it as $x \in \overline{\mathcal{O}^+(y)}$) and prop 2.1.3 (we want $x \in \overline{\mathcal{O}^+(x)}$) it is enough to show that $x \in \overline{\mathcal{O}^+(y)}$. Let U be an open neighbourhood of x. Let us consider two cases.

Case 1: $y \in \mathcal{O}^+(x)$. Then $y = T^n(x)$, for some $n \in \mathbb{N}$. Since x is almost periodic, there is an integer $m > n$, s.t. $T^m(x) \in U$ (def 2.1.8). Therefore $T^{m-n}(y) \in U$, so $x \in \overline{\mathcal{O}^+(y)}$ ($m - n > 0$).

Case 2: $y \in \overline{\mathcal{O}^+(x)} \setminus \mathcal{O}^+(x)$. Let $\varepsilon > 0$ be small enough s.t. $B(x, 2\varepsilon) \subset U$. Such an ε exists since U is open. Since $y \in \overline{\mathcal{O}^+(x)} \setminus \mathcal{O}^+(x)$, there is a sequence $n_j \xrightarrow{j \to \infty} \infty$

such that

$$T^{n_j}(x) \longrightarrow y.$$

Since x is almost periodic and $B(x,\varepsilon)$ is open, there is an integer $K>0$, s.t. for all $j \in \mathbb{N}$, there is $b_j \in \{0,1,\cdots,K\}$ satisfying

$$T^{n_j+b_j}(x) \in B(x,\varepsilon). \quad \text{(dy relolense)}$$

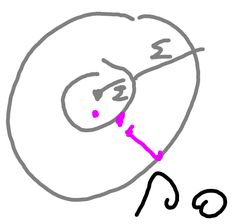
Since X is compact, T^b is uniformly continuous $\forall b \in \{0,\cdots,K\}$.

Therfore we may choose $\delta>0$, s.t. we have

$$d(z,w)<\delta \Rightarrow d(T^b(z),T^b(w))<\varepsilon \quad (*)$$

$\forall b \in \{0,\cdots,K\}$ simultaneously. Recall that $T^{n_j}(x) \longrightarrow y$, so we may choose j large enough, s.t. $d(T^{n_j}(x),y)<\delta$. Then by $(*)$ we have

$$d(T^{b_j+n_j}(x), T^{b_j}(y)) < \varepsilon,$$



So in particular $T^{b_j}(y) \in B(x,2\varepsilon) \subset U$, which shows that $x \in \overline{O^+(y)}$. (x is in closure als el^t van set zonder closure in neighborhood van x ligt!)

$\Leftarrow$ Assume that x is not almost periodic. Then by definition, there exists some open neighbourhood U of x, s.t. the set $\{i \in \mathbb{N} \mid T^i(x) \in U\}$ is not relatively dense. Thus there are sequences n_j and b_j, with $b_j \xrightarrow{j\to\infty} \infty$

satisfying $T^{n_j+b}(x) \notin U$ for any $b \in \{0,1,\dots,b_j\}$.

Since $\overline{\mathcal{O}^+(x)}$ is compact, we may choose a subsequence of n_j denoted by n_{j_i}, s.t. $T^{n_{j_i}}(x) \xrightarrow{i\to\infty} y \in \overline{\mathcal{O}^+(x)}$. Fix $b \in \mathbb{N}$.

By the continuity of T, we have $T^{n_{j_i}+b}(x) \xrightarrow{i\to\infty} T^b(y)$. Now for all i large enough, s.t. $b \le b_j$, we have $T^{n_{j_i}+b}(x) \notin U$, ? so $T^b(y) \notin U$?. Since this is true for any $b \in \mathbb{N}$, we see that $x \notin \overline{\mathcal{O}^+(y)}$, which by remark 2.1.7 implies that $\mathcal{O}^+(x)$ is not minimal. (wer da map!)

QED

2.1.11 Example Let $R_\alpha : S^1 \to S^1$ be the rotation by angle $2\pi\alpha$, where $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. Since the orbit of every point is dense (↑ D 1.1.8(ii)), R_α is minimal (↑ D 2.1.7). Thus all points are recurrent, non-wandering and almost periodic. ! The map $E_m : S^1 \to S^1$ has a dense orbit, but is not minimal, since it has periodic points (↳ Zeige b Zeige ab $x = E_m^b(x) = m^b x \bmod 1$). Every point is non wandering, but not all points are recurrent.

2.2 Topological transitivity

2.2.1 Definition A topological dynamical system

(X,T) is topologically transitive if for all open sets $U,V \neq \emptyset$ there exists $n \in \mathbb{N}$, s.t.

$$T^n(U) \cap V \neq \emptyset.$$

2.2.2 Remark In some references, a system (X,T) is called topologically transitive if it has a dense orbit. In general this is independent of our definition 2.2.1. Indeed, Let $X = \mathbb{N}$ and $T(x) = x+1$. Then $\mathcal{O}^+(0) = X$, so the system (X,T) has a dense orbit, but letting $U = \{1\}$ and

To orbit van 0 is vb van dense orbit

$V = \{0\}$ which are open in the discrete topology, we see that

$$T^n(U) \cap V = \emptyset,$$

$\forall n \in \mathbb{N}$, so (X,T) is not topologically transitive by our definition.

An example of a system which is topologically transitive but does not have a dense orbit is given by

$$X = \{y \in S^1 \mid y \text{ is periodic under } E_2\},$$
$$T = E_2|_X.$$

The proof is left as an exercise to the reader.

Despite the definitions not being equivalent in full generality, the following theorem shows that they are in fact equivalent in a large class of dynamical systems.

2.2.3 Theorem Let X be a complete seperable metric space (or a locally compact Hausdorff space) which does not contain isolated points and let $T: X \to X$ be a continuous map. Then (X,T) is topologically transitive $\Leftrightarrow$ it has a dense orbit.

proof: $\Rightarrow$ Let $V \subset X$ be open. Since T is continuous, $T^{-n}(V)$ is open for all $n \in \mathbb{N}$ and consequently the set

$$\bigcup_{n \in \mathbb{N}} T^{-n}(V) \qquad (1)$$

is open. Moreover, since the system (X,T) is topologically transitive, for any open $U \subset X$ there is an integer n s.t. $T^n(U) \cap V \neq \emptyset$, which readily implies that the set in (1) is dense. ?

Let $\{V_i\}_{i \in \mathbb{N}}$ be a countable basis of the topology of X, which exists since X is a complete seperable metric space. By the previous discussion, each of the sets

$\bigcup_{n\in\mathbb{N}} T^{-n}(V_i)$, are open and dense, so by the Baire category theorem, the set

$$\bigcap_{i\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} T^{-n}(V_i),$$

is dense. In particular, it is non-empty, so taking $y \in \bigcap_{i\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} T^{-n}(V_i)$, we have

$$\mathcal{O}^+(y) \cap V_i \neq \emptyset, \quad ?$$

$\forall i \in \mathbb{N}$. Since $\{V_i\}_{i\in\mathbb{N}}$ is a basis of the topology of X, this shows that for any open $U \subset X$, we have

$$\overline{\mathcal{O}^+(y)} \cap U \neq \emptyset,$$

so the orbit of y is dense.

$\Leftarrow$ Assume that $\overline{\mathcal{O}^+(x)} \overset{\text{dense}}{=} X$. Let $U, V \neq \emptyset$ be open sets. Then there exists $m \in \mathbb{N}$ s.t. $T^m(x) \in U$?. Since X has no isolated points, the set

$$V \setminus \bigcup_{i=0}^{m} \{T^i(x)\}$$

is non empty and open?. Since the orbit of x is dense, there exists $n > m$ s.t. $T^n(x) \in V$. Now

$$T^{n-m}(\underbrace{T^m(x)}_{\in U}) \in V, \text{ giving } T^{n-m}(U) \cap V \neq \emptyset.$$

QED ■

2.2.4 Remark Let $T: X \to X$ be a homeomorphism.

It is possible that there exists $x \in X$ with $\overline{\mathcal{O}(x)} = X$, but $\overline{\mathcal{O}^+(y)} \neq X \ \forall y \in X$. For example, let $X = \mathbb{Z}$ and let $T : X \to X$, $T(x) = x+1$.

2.2.5 Theorem Let X be a complete separable metric space and let $T : X \to X$ be a homeomorphism. Assume that X has no isolated points. Then if there is a dense orbit $\overline{\mathcal{O}(x)}$, there is a dense forward orbit $\overline{\mathcal{O}^+(y)}$
(i.e. $\overline{\mathcal{O}(x)} = X$)

Proof: Exercise

2.2.6 Theorem If T is topologically transitive, then every forward invariant open set is dense.

proof: Let T be topologically transitive and let $U \neq \emptyset$ an open set s.t. $T(U) \subset U$ (T is forward invariant).
(as via CD)
If U is not dense, there exists an open set $V \neq \emptyset$ s.t. $U \cap V = \emptyset$. Since $T(U) \subset U$, we have $T^n(U) \subset U$ $\forall n \in \mathbb{N}$. Then of course $T^n(U) \cap V = \emptyset \ \forall n \in \mathbb{N}$ (T is topologically transitive) ↯

QED ■

2.2.7 Remark If T is an open map (for example, a homeomorphism), then the converse of Theorem 2.2.6 is true.

proof: Let T be an open map. If T is not topologically (→ via CP) transitive, there are non-empty open sets U and V s.t.

(*) $T^n(U) \cap V = \emptyset$ $\forall n \in \mathbb{N}$. Let (Zo'n W, zodat W inv die niet dense is)

$$W = \bigcup_{n=0}^{\infty} T^n(U).$$

Then $W \neq \emptyset$ (→ $U \neq \emptyset$) is open (→ unie van open sets!) and

apply T at both sides: $T(W) = \bigcup_{n=1}^{\infty} T^n(U) \subset W,$ (overbodig, alleen $T^0(U)$ is er niet)

so W is forward invariant. But by (*) $W \cap V = \emptyset$, hence W is not dense.

QED ■

2.3 Topological mixing

Definition 2.3.1 A topological dynamical system (X, T) is *topologically mixing*, if for all open sets $U, V \neq \emptyset$ there exists $N \in \mathbb{N}$ s.t.

$$T^n(U) \cap V \neq \emptyset$$

$\forall n \geq N$.

2.3.2 Remark Clearly, a topologically mixing

system is topologically transitive, but the converse is not necessarily true. For example, the irrational rotation $R_\alpha : S^1 \to S^1$ is topologically transitive (E.). It is, however not topologically mixing. To see this, let $U =]0, \frac{1}{10}[$ and $V =]\frac{1}{2}, \frac{6}{10}[$. Then for every $n \in \mathbb{N}$, the set $R_\alpha^n(U)$ is an open interval with length $\frac{1}{10}$ and having the point $n\alpha \bmod 1$ as an endpoint. Thus there are arbitrarily large n s.t. $R_\alpha^n(U) \cap V = \emptyset$.

2.3.3 Proposition Let X be a metric space and Let $T : X \to X$ be an isometry. Then T is not topologically mixing provided that X contains at least two points.

proof: Let $x, y \in X$ be distinct points and let $\varepsilon = \frac{1}{5} d(x,y) > 0$.

It suffices to show that, for every $n \in \mathbb{N}$, the condition

$$T^n(B(x,\varepsilon)) \cap B(y,\varepsilon) \neq \emptyset$$

?

implies that

$$T^n(B(x,\varepsilon)) \cap B(x,\varepsilon) = \emptyset.$$

To see that this is the case, suppose $\exists z \in T^n(B(x,\varepsilon)) \cap B(y,\varepsilon)$ and $w \in T^n(B(x,\varepsilon)) \cap B(x,\varepsilon)$. Then since T is an isometry,

provided

$$d(x,y) \overset{\Delta\text{-ineq}}{\leq} d(x,w) + d(w,z) + d(z,x) \leq \varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon < d(x,y) \lightning$$

$w, z \in T^n(B(x,\varepsilon))$

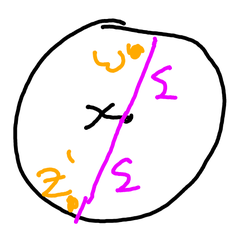
$d(T^n(w'), T^n(z'))$ for $w', z' \in B(x,\varepsilon)$

$\parallel$ isom

$d(w', z')$

? isom. ?

QED ∎



$dist \leq 2\varepsilon$

Example 2.3.4 The map $A: T^2 \to T^2$ in example 1.3.7 is topologically mixing.

proof: By problem 4 of exercise 2 we know that the unstable manifold $W^u(x)$ is dense in T^2 for all $x \in T^2$. Thus, for all $\varepsilon > 0$, the collection $\{B(y,\varepsilon)\}_{y \in W^u(x)}$ is an open cover of T^2. Since T^2 is compact, there exists a finite subcover. Thus there exists an interval $S \subset W^u(x)$ with finite length which is ε-dense in T^2, that is, $d(y,S) < \varepsilon$ for all $y \in T^2$. Since a translation on T^2 is an isometry, $z+S$ is ε-dense in T^2 for all $z \in T^2$.

observe that $z+S \subset W^u(z)$. Therefore, for any $\varepsilon > 0$, there is $0 < L(\varepsilon) < \infty$ s.t. every interval of length at least $L(\varepsilon)$ on any unstable manifold $W^u(z)$ is ε-dense in T^2.

Let $U, V \neq \emptyset$ be open. Let $y \in V$ and let $\varepsilon > 0$ s.t. $\overline{B(y,\varepsilon)} \subset V$.

Now $U \cap W^u(x)$ contains an interval of length δ for some $\delta > 0$ and $x \in T^2$.

Let $N \in \mathbb{N}$ be s.t. $d^N \delta > L(\Sigma)$. Then for every $n \geq N$ the set $A^n(U)$ contains an interval J of length $L(\Sigma)$ on $W^u(A^n(x))$. Since J is Σ-dense, $d(y, J) < \Sigma$ and $A^n(U) \cap V \neq \emptyset$ $\forall n \geq N$.

QED

2.3.5 Proposition The systems (Σ_m, σ) and (Σ_m^+, σ) are topologically mixing

proof: We only prove it for (Σ_m, σ). Note that the topology of Σ_m is generated by the open balls

$$B(x, m^{-\ell}) := \{ y \in \Sigma_m \mid y_{-1} = x_{-1}, \dots, y_\ell = x_\ell \},$$?

so it is enough to show that for any balls $B(x, m^{-\ell})$, $B(y, m^{-b})$ there exists $N \in \mathbb{N}$, s.t.

$$\sigma^n(B(x, m^{-\ell}) \cap B(y, m^{-b})) \neq \emptyset,$$

$\forall n \geq N$. Fix such balls and let $N = \ell + b + 1$. Let $n \geq N$ and note that for

$$z := (\dots, \underbrace{x_{-\ell}, \dots, x_0}_{\text{"x gespiegelt"}}, \overbrace{\dots, x_\ell}^{\ell+1}, \overbrace{z_{\ell+1}, \dots, z_{n-b-1}}^{n-b-1}, \overbrace{y_{-b}, \dots, y_0, \dots, y_b}^{b+1}, \dots),$$

where $z_0 = x_0$ and the letters which are not specified may be

Chosen arbitrarely, we have $z \in B(x, m^{-\ell})$. Now

$$\sigma^n z = (\dots, x_{-\ell}, \dots, x_\ell, z_{\ell+1}, \dots, z_{n-b-1}, y_{-b}, \dots, y_0, \dots, y_b, \dots),$$

where $(\sigma^n z)_0 = y_0$

$\ell + 0 + b = n + \ell + 1$ IDK ;;;

So in particular $\sigma^n z \in B(y, m^{-b})$, this implies that

$$\sigma^n(B(x, m^{-\ell})) \cap B(y, m^{-b}) \neq \emptyset,$$

$\forall n \geq N$!?

QED ∎

2.3.6 Remark We note that if (X, T) is mixing and there is a topological semiconjugacy $\pi: X \to Y$, then (Y, S) is also topologically mixing (E). Thus the previous proposition has the following immediate corollary.

2.3.7 Corollary The system (S^1, E_m) of example 1.2.4 is topologically mixing.

Proof: The topological semiconjugacy is given by proposition 1.3.3. Now it follows by Proposition 2.3.5. QED ∎

2.3.8 Example The solenoid (S, T) of Example 1.3.10(c) is topologically mixing.

Proof: Let $\Phi \subset (S^1)^{\mathbb{N}}$ defined by

$$\Phi = \{ \phi \in (S^1)^{\mathbb{N}} \mid \phi_i = 2\phi_{i+1} \mod 1, \forall i \in \mathbb{N} \},$$

and equip the space Φ with the subs :

Dit lijkt me onbelangrijk

⋮

2.4 Expansiveness

2.4.1 Definition A homeomorphism $T: X \to X$ is expansive if there exists $\delta > 0$, s.t. $\forall x, y \in X$, $x \neq y$, there exists $n \in \mathbb{N}$, s.t.

$$d(T^n(x), T^n(y)) \geq \delta.$$

A continuous map $T: X \to X$ is forward expansive if this property holds for some $n \in \mathbb{N}$. The constant δ is called the expansiveness constant of T.

2.4.2 Example The maps $E_m: S^1 \to S^1$, $\sigma: \Sigma_m^{(+)} \to \Sigma_m^{(+)}$, $A: \mathbb{T}^2 \to \mathbb{T}^2$ and the solenoid are expansive. The rotation $R_2: S^1 \to S^1$ and more generally any isometry is not expansive, assuming that for all $\delta > 0$ there are distinct points x, y s.t. $d(x, y) < \delta$.

At first glance it might feel natural to require the property for expansiveness to hold for positive integers also in the case of a homeomorphism. However, the next theorem shows that this does not give a meaningfull definition.

2.4.3 Theorem Let X be an infinite compact metric space and $T: X \to X$ a homeomorphism. Then for all $\varepsilon > 0$ there are distinct points $x, y \in X$, s.t. $d(T^n(x), T^n(y)) < \varepsilon$ $\forall n \in \mathbb{N}$.

Proof: Fix $\varepsilon > 0$ and let

$$E := \{n \in \mathbb{N} \mid \exists x, y \in X, d(x,y) \geq \varepsilon, d(T^i(x), T^i(y)) < \varepsilon, \forall i = 1, \dots, n\}.$$

Let $M = \sup E$ if $E \neq \emptyset$ and otherwise $M = 0$.

If $M = \infty$, then for all $n \in \mathbb{N}$, there are $x_n, y_n \in X$, s.t. the condition defining E is satisfied. By compactness, we may extract subsequences s.t.

$$\lim_{k \to \infty} x_{n_k} = x', \text{ and } \lim_{k \to \infty} y_{n_k} = y'.$$

Since $d(x_{n_k}, y_{n_k}) \geq \varepsilon$ (def E), we have $d(x', y') \geq \varepsilon$, and using the continuity of the metric d and the mapping T, we have

$$d(T^n(x'), T^n(y')) = \lim_{k \to \infty} d(T^n(x_{n_k}), T^n(y_{n_k})) < \varepsilon \quad (\text{def } E),$$

for all $n \in \mathbb{N}$. Then the points $x = T(x')$ and $y = T(y')$ satisfy the claim.

— warum nicht x' und y' selbst?

If $M < \infty$. Since the family $\{T, T^2, \dots, T^M\}$ is equicontinuous (since X is compact, then exists $0 < \delta < \varepsilon$ s.t. the condition $d(x,y) < \delta$ implies that $d(T^n(x), T^n(y)) < \varepsilon$

for all $n \in \{0,1,\dots,M\}$. By definition of M we have that

$$d(T^{-1}(x), T^{-1}(y)) < \varepsilon,$$

and by induction we obtain that

$$d(T^{-n}(x), T^{-n}(y)) < \varepsilon,$$

$\forall n \in \mathbb{N}$. Since X is compact, $\exists$ a collection $\mathcal{B}$ of k balls with radius $\delta/2$ which is a cover of X. Let $W \subset X$ be any set with $k+1$ el^{ts}. Then for every $j \in \mathbb{Z}$ there exist $a_j, b_j \in W$ s.t. $T^j(a_j), T^j(b_j) \in \mathcal{B}$ and $a_j \neq b_j$. Therefore,

$$d(T^n(a_j), T^n(b_j)) < \varepsilon$$

$\forall n \leq j$. Since W is finite, there exist $x_0, y_0 \in W$ s.t. $a_j = x_0$ and $b_j = y_0$ for infinitely many $j \in \mathbb{N}$. Thus $d(T^n(x_0), T^n(y_0)) < \varepsilon$ $\forall n \in \mathbb{N}$. QED ∎

2.4.4 Corollary Let X be an infinite compact metric space and let $T: X \to X$ be an expansive homeomorphism. Then there exist distinct points $x, y \in X$ s.t.

$$\lim_{n \to \infty} d(T^n(x), T^n(y)) = 0$$

Proof: Let δ be an expansive constant of T. By Theorem 2.4.3 there are distinct points $x, y \in X$ s.t.

$$\phi(T^n(x), T^n(y)) < \frac{\delta}{2}$$

$\forall n \in \mathbb{N}$. Suppose that $d(T^n(x), T^n(y)) \not\to 0$ as $n \to \infty$. (by < 0)

Then we can choose a subsequence (n_k) s.t. $\lim_{k\to\infty} T^{n_k}(x) = x_0$,

$\lim_{k\to\infty} T^{n_k}(y) = y_0$ and $y_0 \neq x_0$. Now for every $m \in \mathbb{Z}$ we have that

$$\lim_{k\to\infty} T^{n_k+m}(x) = T^m(x_0) \quad \text{and} \quad \lim_{k\to\infty} T^{n_k+m}(y) = T^m(y_0).$$

For any $m \in \mathbb{Z}$ it is true that $n_k + m > 0$ for all large k.

Therefore $n_k + m \in \mathbb{N} \Rightarrow d(T^{n_k+m}(x), T^{n_k+m}(y)) < \frac{\delta}{2}$ (to be $\lim$) $\Rightarrow$

$$d(T^m(x_0), T^m(y_0)) \leq \frac{\delta}{2} < \delta \quad ?$$

($\leq$ becomes $<$ after taking $\lim$.)

$\forall m \in \mathbb{Z}$!, which is a contradiction with expansiveness.

QED

2.4.5 Remark

a) Assume that X is a compact topological space and d_1 and d_2 are two metrics which induce the topology of X. Then the map $T:(X, d_1) \to (X, d_1)$ is expansive $\Leftrightarrow$ $T:(X, d_2) \to (X, d_2)$ is expansive (E).

b) If X and Y are compact, (X, T) is expansive and (Y, S) is conjugate to (X, T), then (Y, S) is expansive (E).

2.5 Topological entropy

Topological entropy is a topological invariant which describes the complexity of the orbit structure.

2.5.1 Definition Let (X,T) be a topological dynamical system and let (X,d) be a compact metric space. For all $n \in \mathbb{N}$ and $x, y \in X$, set

$$d_n(x,y) = \max_{0 \le k < n} d(T^k(x), T^k(y)).$$

2.5.2 Remark For all $n \in \mathbb{N}$, the map d_n is a metric on X, $d_{n+1} \ge d_n$, $d_0 = d$ and they all induce the same topology. (E)

2.5.3 Definition Let X, T and d be as in Definition 2.5.1 and let $\varepsilon > 0$. A set $A \subset X$ is (n,ε)-spanning, if for every $x \in X$ there exists $a \in A$ s.t.

$$d_n(x,a) < \varepsilon$$

Let

$$\mathrm{span}(n,\varepsilon,T) := \min\{\#A \mid A \subset X \text{ is } (n,\varepsilon)\text{-spanning}\}$$

A set $A \subset X$ is (n,ε)-separated, if

$$d_n(x,y) \ge \varepsilon \quad \forall\, x,y \in A \text{ with } x \neq y.$$

Let

$$\text{sep}(n,\varepsilon,T) = \max\{\#A \mid A \subset X \text{ is } (n,\varepsilon)\text{-separated}\}$$

↳ 1 punt always sep.

Finally, define

$$\text{cov}(n,\varepsilon,T) = \min\left\{\#\mathcal{A} \mid X \subset \bigcup_{A\in\mathcal{A}} A,\ \text{diam}_n(A) < \varepsilon\ \forall A\in\mathcal{A}\right\}$$

↳ alle covers met diameter $< \varepsilon$

where diam_n denotes the diameter in the metric d_n.

2.5.4 Remark Since X is compact, the numbers $\text{span}(n,\varepsilon,T)$, $\text{sep}(n,\varepsilon,T)$ and $\text{cov}(n,\varepsilon,T)$ are well defined and finite.

2.5.5 Lemma Let X, T and d be as in Definition 2.5.1. Then for every $\varepsilon > 0$ and $n \in \mathbb{N}$ it is true that

$$\text{cov}(n,2\varepsilon,T) \leq \text{span}(n,\varepsilon,T) \leq \text{sep}(n,\varepsilon,T) \leq \text{cov}(n,\varepsilon,T).$$

Proof: We will first prove the first inequality. Let A be an (n,ε)-spanning set. Then

$$X \subset \bigcup_{a\in A} B_n(a,\varepsilon),$$ (ergens logisch door def (n,ε)-span)

where B_n denotes the open ball in the metric d_n. Now there exists $\varepsilon_1 < \varepsilon$ s.t.

$$X \subset \bigcup_{a\in A} B_n(a,\varepsilon_1).$$

Indeed, since $B_n(a,\varepsilon) = \bigcup_{b\in\mathbb{N}} B_n(a,\varepsilon - \frac{1}{b})$, we have that

$$X \subset \bigcup_{b\in\mathbb{N}} \bigcup_{a\in A} B_n(a, \varepsilon - \tfrac{1}{b}).$$

Since X is compact, this open cover has a finite subcover. Now

writing ε_1 for $\varepsilon - \frac{1}{b}$ — $\to 0$

$$\operatorname{diam}_n B_n(a, \varepsilon_1) \leq 2\varepsilon_1 < 2\varepsilon,$$

hence $\operatorname{cov}(n, 2\varepsilon, T) \overset{?}{\leq} \operatorname{span}(n, \varepsilon, T)$. Proof of the inequality $\operatorname{span}(n, \varepsilon, T) \leq \operatorname{sep}(n, \varepsilon, T)$ is left as an exercise.

To see that the final inequality holds, let $\{A_i\}$ be a minimal (n, ε)-cover. Then every (n, ε)-separated set contains at most one point in each set A_i? Therefore,

$$\operatorname{sep}(n, \varepsilon, T) \leq \operatorname{cov}(n, \varepsilon, T) \qquad \text{QED} \blacksquare$$

2.5.6 Definition. The topological entropy of T is

$$h_{top}(T) = \lim_{\varepsilon \to 0^+} h_\varepsilon(T)$$

where

↦ limsup (lim may not exist)

$$h_\varepsilon(T) = \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{cov}(n, \varepsilon, T).$$

↦ Lemma 2.5.5

Remark 2.5.7 Since $\operatorname{cov}(n, \varepsilon, T)$ increases as ε decreases, we have that

$$h_{top}(T) = \lim_{\varepsilon \to 0^+} h_\varepsilon(T) = \sup_{\varepsilon > 0} h_\varepsilon(T).$$

By Lemma 2.5.5 we may replace $\operatorname{cov}(n, \varepsilon, T)$ by

$\text{span}(n,\varepsilon,T)$ or $\text{sep}(n,\varepsilon,T)$ in the definition of h_{top}.

Since $\text{cov}(n,\varepsilon,T) \geq 1$, we have that

$$h_{top}(T) \geq h_\varepsilon(T) \geq 0.$$

en $\frac{1}{n}\cdot\text{cov}$

1e gelijkheid

2.5.8 Proposition The limit

$$\lim_{n\to\infty} \frac{1}{n} \log \text{cov}(n,\varepsilon,T) = h_\varepsilon(T)$$

exists and is finite.

? Proof: First observe that if $\text{diam}_m(U) < \varepsilon$ and $\text{diam}_n(V) < \varepsilon$, then

$$\text{diam}_{n+m}(U \cap T^{-m}(V)) < \varepsilon,$$

where $\text{diam}_n(U)$ denotes the diameter of U w.r.t. the metric d_n. Also note that if $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ are covers of X, then the collection $\{U_i \cap T^{-m}(V_j)\}_{(i,j)\in I\times J}$ covers X. Therefore it is clear that

$$\text{cov}(n+m,\varepsilon,T) \leq \text{cov}(n,\varepsilon,T)\,\text{cov}(m,\varepsilon,T),$$

so in particular, the sequence $\log \text{cov}(n,\varepsilon,T)$ is subadditive. It is an easy exercise to show that if a_n is a subadditive sequence, then the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists and is equal to $\inf_{n\in\mathbb{N}} \frac{a_n}{n}$, which gives that

$$\lim_{n\to\infty} \frac{1}{n} \log \text{cov}(n,\varepsilon,T) = \inf_{n\in\mathbb{N}} \frac{1}{n} \log \text{cov}(n,\varepsilon,T) < \infty$$

QED ∎

2.5.9 Remark By Lemma 2.5.5 and proposition 2.5.8, we have that

$$h_{top}(T) = \lim_{\varepsilon \to 0^+} \liminf_{n\to\infty} \frac{1}{n} \log span(n, \varepsilon, T)$$
$$= \lim_{\varepsilon \to 0^+} \liminf_{n\to\infty} \frac{1}{n} \log sep(n, \varepsilon, T)$$

2.5.10 Proposition Let X be a compact metric space and let $T: X \to X$ be an isometry. Then

$$h_{top}(X) = 0.$$

Proof: Since for all $n \in \mathbb{N}$ and $x, y \in X$ we have $d(T^n(x), T^n(y)) \overset{\text{isom.}}{=} d(x,y)$, we have $d_n = d$. Therefore $cov(n, \varepsilon, T) \overset{d_n = d,\ \text{def cov}}{=} cov(0, \varepsilon, T)$ $\forall n \in \mathbb{N}$ and thus

$$h_{top}(T) = \lim_{\varepsilon \to 0^+} \underbrace{\lim_{n\to\infty} \frac{1}{n} \log cov(0, \varepsilon, T)}_{=0} = 0$$

2.5.11 Theorem Let X be a compact topological space and let d and d' be metrics inducing the topology of X. The d and d' give the same topological entropy for all continuous maps $T: X \to X$.

Proof: For all $\varepsilon > 0$, define

$$\delta(\varepsilon) := \sup \{ d'(x,y) \mid d(x,y) \leq \varepsilon \}$$

Since d and d' induce the same topology on X, $\forall x \in X$,

$\exists\, \varepsilon > 0$ $d' \leq d \leq \varepsilon$

there is a finite number

$$\delta(\varepsilon, x) = \sup_{y \in X} \{ d'(x,y) \mid d(x,y) \leq \varepsilon \},$$

$$\vdots$$

2.5.12 Corollary Topological entropy is a topological invariant (on compact topological spaces).

if X homeo. with Y and P is a property of ⇒ P(X) = P(Y) or conj. ⇒ so also a homeo

Proof: Let $\pi : (X, T) \to (Y, S)$ be a topological conjugacy.

Let d be a metric on Y, which induces the topology of Y.

Set $d'(x_1, x_2) \overset{*}{=} d(\pi(x_1), \pi(x_2)) \quad \forall x_1, x_2 \in X$.

Then d' is a metric on X, which generates the topology of X (E!)

By Theorem 2.5.11,

$$h'_{top}(T) = h_{top}.$$

On the other hand, $\pi : (X, d') \to (Y, d)$ is an ⌜isometric conjugacy, so combining this with the previous, we have that

$$h_{top}(T) = h_{top}(S) \qquad (E). \qquad \text{QED}$$

2.5.13 Corollary Let (X, T) and (Y, S) be topological dynamical systems, where X and Y are compact. If $h_{top}(T) \neq h_{top}(S) \Rightarrow (X, T)$ and (Y, S) are not topologically conjugate. (CP v. 2.5.12)

2.5.14 Theorem Let X be a compact metric space

and $T: X \to X$ continuous, then the following hold.

a) $h_{top}(T^m) = m \cdot h_{top}(T) \quad \forall m \in \mathbb{N}$.

b) If T is invertible, then

$$h_{top}(T^{-1}) = h_{top}(T).$$

In particular

$$h_{top}(T^m) = |m| \, h_{top}(T), \quad \forall m \in \mathbb{Z}.$$

c) If $A_i \subset X$, $i = 1, \dots, b$ are closed, forward invariant sets, which cover X, then

$$h_{top}(T) = \max_{i=1,\dots,b} h_{top}(T|_{A_i}).$$

In particular, if A is a closed, forward invariant set, then

$$h_{top}(T|_A) \leq h_{top}(T).$$

Proof: a) We proof two inequalities.

$\leq$ Since $\max_{0 \leq i \leq n} d(T^{m \cdot i}(x), T^{m \cdot i}(y)) \leq \max_{0 \leq j \leq m \cdot n} (d(T^j(x), T^j(y))$,

we have that $span(n, \varepsilon, T^m) \overset{\text{def span}}{\leq} span(m \cdot n, \varepsilon, T)$ (↪ alle veelvouden). Thus

$$\limsup_{n \to \infty} \frac{1}{n} \log span(n, \varepsilon, T^m) \overset{\cdot \frac{m}{m} \leq 1}{\leq} \limsup_{n \to \infty} \frac{m}{mn} \log span(mn, \varepsilon, T)$$

$$\leq m \cdot \limsup_{j \to \infty} \frac{1}{j} \log span(j, \varepsilon, T).$$

Taking the limit $\varepsilon \to 0^+$ on both sides, we obtain the inequality $h_{top}(T^m) \overset{\text{def } h_{top}}{\leq} m \cdot h_{top}(T)$.

anisometric ∇

≥) Since X is compact, $\forall \varepsilon > 0$, $\exists\ \delta(\varepsilon) > 0$ s.t. the condition
↳ so T uniform. cont.

$d(x,y) < \delta(\varepsilon)$ implies that $d(T^i(x), T^i(y)) < \varepsilon\ \ \forall i = 1, \dots, m$.

Therefore,

$$span(n, \delta(\varepsilon), T^m) \geq span(mn, \varepsilon, T) \qquad ?$$

Thus,

$$\liminf_{n\to\infty} \frac{1}{n} \log span(n, \delta(\varepsilon), T^m) \geq \liminf_{n\to\infty} \frac{m}{m\cdot n} \log span(m\cdot n, \varepsilon, T) \qquad \cdot\frac{m}{m}$$

$$\geq m \cdot \liminf_{j\to\infty} \frac{1}{j} span(j, \varepsilon, T).$$

letting $\varepsilon \to 0$ (so that also $\delta(\varepsilon) \to 0$), Remark 2.5.9 implies that $h_{top}(T^m) \geq m \cdot h_{top}(T)$. ?

b) The claim follows from the observation if a set A is (n,ε)-separated for T, then $T^m(A)$ is (n,ε)-separated for T^{-1} and vice versa.

$d_n(x,y) \geq \varepsilon \quad \forall x, y \in A$

$\| \max_{0\leq k\leq m} (d(T^k(x), T^k(y)) \geq \varepsilon$

$d_n(T^m(x), T^m(y)) = \max_{0\leq k\leq m} (d(T^{-k}T^m(x), T^{-k}T^m(y)) \geq \varepsilon$

$= \max\{d(T^m(x), T^{m-k}(y)), d(T^{m-k}(x), T^{m-k}(y)), \dots, d(x,y)\}$

? c) By assumption $X \subset \bigcup_{i=1}^{b} A_i$. If a set B is (n,ε)-separated in each A_i, then B is (n,ε)-separated in X.
A_i cover X

Hence $sep(n, \varepsilon, T|_{A_i}) \leq sep(n, \varepsilon, T)\ \ \forall i = 1, \dots, b$. This yields by the inequality

more ineq:

$$\max_{1\leq i\leq b} h_{top}(T|_{A_i}) \leq h_{top}(T).$$

Towards the opposite inequality, observe that if B_i is (n,ε)-spanning in A_i, then $\bigcup_{i=1}^{b} B_i$ is (n,ε)-spanning in X.

Thus,

$$\operatorname{span}(n,\varepsilon,T) \leq \sum_{i=1}^{b} \operatorname{span}(n,\varepsilon,T|_{A_i}) \overset{\text{altijd}}{\leq} b \cdot \max_{1\leq i\leq b} \operatorname{span}(n,\varepsilon,T|_{A_i}).$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{n}\log \operatorname{span}(n,\varepsilon,T)$$

$\log(\cdot) = \log + \log$ $\rightarrow$ $\leq \limsup_{n\to\infty} \frac{\log b}{n} + \limsup_{n\to\infty} \frac{1}{n}\log\left(\max_{1\leq i\leq b} \operatorname{span}(n,\varepsilon,T|_{A_i})\right)$ (limsup(+) $\leq$ limsup + limsup)

$\frac{\log b}{n} \to 0$

$$= \max_{1\leq i\leq b} \limsup_{n\to\infty} \frac{1}{n}\log\left(\operatorname{span}(n,\varepsilon,T|_{A_i})\right).$$

Letting $\varepsilon \to 0^+$, we conclude that $h_{top}(T) \leq \max_{1\leq i\leq b} h_{top}(T|_{A_i})$

QED ∎

2.5.15 Theorem Let (X,d^X), (Y,d^Y) be compact metric spaces and let $T: X\to X$, $S: Y\to Y$ be continuous maps. Then

a) $h_{top}(T\times S) = h_{top}(T) + h_{top}(S)$ and

b) if (Y,S) is a factor of (X,T), then $h_{top}(T) \geq h_{top}(S)$

Proof: a) The maximum metric

$$d((x,y),(x',y')) = \max\{d^X(x,x'), d^Y(y,y')\}$$

generates the product topology on $X\times Y$ and

$$d_n((x,y),(x',y')) \overset{\text{def max meth.}}{=} \max\{d_n^X(x,x'), d_n^Y(y,y')\}.$$

V) If $U\subset X$ and $V\subset Y$ are sets with diameter less than ε, then the diameter of $U\times V\subset X\times V$ is less

logisch

than ε. Thus

→ heeft meer covers met diam < ε door wat wer ver zit en zie def cov

$$? \quad cov(n,\varepsilon,T\times S) \leq cov(n,\varepsilon,T)\, cov(n,\varepsilon,S)$$

→ "all possible products"

and we conclude that

$$h_{top}(T\times S) \leq h_{top}(T) + h_{top}(S).$$

(def h_{top} en log map)

$\geq$) If $A\subset X$ and $B\subset Y$ are (n,ε)-seperated, then ?
$A\times B\subset X\times Y$ is (n,ε)-seperated. Thus

zelfde redenering als bij cov

$$sep(n,\varepsilon,T\times S) \geq sep(n,\varepsilon,T)\, sep(n,\varepsilon,S).$$

By Remark 2.5.9 we obtain that

$$h_{top}(T\times S) = h_{top}(T) + h_{top}(S).$$

G) (E)

QED ■

2.5.16 Theorem Let (X,d) be a compact metric space and let $T: X \to X$ be an expansive homeomorphism with expansiveness constant δ. Then

$$h_{top}(T) = h_\varepsilon(T)$$

$\forall \varepsilon < \delta$.

zie def h_ε en cov (meer servolletjes van diam < ε ⇒ tenminste min)

Proof: Let $\gamma < \varepsilon < \delta$. Since $h_\gamma(T) \geq h_\varepsilon(T)$, it is enough to show that $h_\gamma(T) \leq h_\varepsilon(T)$. This follows if we show that $h_{2\gamma}(T) \leq h_\varepsilon(T)$ $\forall \gamma < \frac{\varepsilon}{2}$. Since T is

→ γ wil $< \varepsilon$ ⇒ $\gamma \to 0$?

expansive, $\forall x, y \in X, \exists i \in \mathbb{Z}$ s.t. $d(T^i(x), T^i(y)) \geq \delta > \varepsilon$. (assumption)

Since the set

$$\{(x,y) \in X \times X \mid d(x,y) \geq \gamma\}$$

is compact, (↩ X comp. & d cont.) there exists $B \in \mathbb{N}$ s.t. the condition $d(x,y) \geq \gamma$ implies that $d(T^i(x), T^i(y)) \geq \varepsilon$ for some $|i| \leq B$. ? If A is (n,γ)-separated, then $T^{-B}(A)$ is $(n+2B, \varepsilon)$-separated ?

$$A \text{ } (n,\gamma)\text{-sep} \Rightarrow d_n(x,y) = \max_{0 \leq j \leq n} \{d(T^j(x), T^j(y))\} \geq \varepsilon$$

$$d_{n+2B}(T^{-B}(x), T^{-B}(y)) = \max_{0 \leq j \leq n+2B} \{d(T^j(T^{-B}(x)), T^j(T^{-B}(y)))\} \quad ?$$
$$T^{n+B}(x), T^{n+B}(y)$$

!

By Lemma 2.5.5,

$$h_\varepsilon(T) = \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{cov}(n, \varepsilon, T)$$

$$\overset{2.5.5}{\geq} \limsup_{n \to \infty} \frac{1}{n} \log(\mathrm{sep}(n, \varepsilon, T))$$

?

$$\geq \limsup_{n \to \infty} \frac{1}{n} \log(\mathrm{sep}(n-2B, \gamma, T))$$

$$\overset{2.5.5}{\geq} \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{cov}(n-2B, 2\gamma, T)$$

herschrijf $= \limsup_{n \to \infty} \frac{1}{(n-2B)\left(1 + \frac{2B}{n-2B}\right)} \log \mathrm{cov}(n-2B, 2\gamma, T)$ (dus $n-2B \to \infty$ ✓) $\quad = \frac{1}{n}$

$$= h_{2\gamma}(T).$$

QED ∎

2.5.17 Remark The entropy of a flow can be defined using the map T^1 or by using the metrics d_t, $t > 0$. These give the same definition since the family T^t, $t \in [0,1]$ is equicontinuous.

Example 2.5.18 Let $A: T^2 \to T^2$ be the map from example 1.3.7. Then $h_{top}(A) = \log d$, where d is the larger eigenvalue of A.

make two inequalities

proof: Let $v_d, v_{1/d}$ be the normalised eigenvectors, which are perpendicular to eachother. Since A expands in direction v_d by the factor d and contracts in direction $v_{1/d}$ by the factor $1/d$ we have for small $\varepsilon > 0$, that

$A v_d = d v$

$= (x_1, x_2) \in T^2$

$B_n(x, \varepsilon) \subset]x_1 - \varepsilon d^{-n}, x_1 + \varepsilon d^{-n}[\times]x_2 - \varepsilon, x_2 + \varepsilon[$, ?

$=: A_n$

in the $v_d v_{1/d}$-plane using the euclidian metric. Note that this induces the standard topology on T^2? so by Theorem 2.5.11 we may use the covering number w.r.t. this metric to calculate the entropy. Now

2e form.: $\mathcal{L}$ of rectangle

$$\mathcal{L}(B_n(x, \varepsilon)) \leq 2\varepsilon d^{-n} \cdot 2\varepsilon = 4\varepsilon^2 d^{-n},$$

Lowaaom niet $= 1$? $\subset$ niet $= \supset$

where $\mathcal{L}$ denotes the two dimensional Lebesgue measure on the torus. Let $\{B_n(x_i, \varepsilon)\}_{i=1}^N$ be a cover of T^2. Then

$\Rightarrow T^2 = \cup \{ \}$

$$1 = \mathcal{L}(T^2) \leq \sum_{i=1}^{N} \mathcal{L}(B_n(x_i, \varepsilon)) \leq N 4\varepsilon^2 d^{-n},$$

so $N \geq (4\varepsilon^2)^{-1} d^n$. This means that we need at least $(4\varepsilon^2)^{-1} d^n$ (n, ε)-balls to cover T^2? Since every set with diameter ε is included in some (n, ε)-ball,

this implies that

$$\mathrm{cov}(n,\varepsilon,A) \geq (4\varepsilon^2)^{-1}\lambda^n. \quad !$$

Thus

$$\begin{aligned} h_{top}(A) &\overset{\text{def}}{=} \lim_{\varepsilon\to 0^+} \lim_{n\to\infty} \frac{1}{n}\log \mathrm{cov}(n,\varepsilon,A) \\ &\geq \lim_{\varepsilon\to 0^+} \lim_{n\to\infty} \frac{1}{n}\log\left((4\varepsilon^2)^{-1}\lambda^n\right) \\ &= \lim_{\varepsilon\to 0^+} \underbrace{\lim_{n\to\infty} \frac{n}{n}\log\lambda}_{\log\lambda} - \lim_{\varepsilon\to 0^+} \underbrace{\lim_{n\to\infty} \frac{1}{n}\log(4\varepsilon^2)}_{0} \\ &= \log\lambda. \end{aligned}$$

on the the hand, we may tile T^2 by the translates of A_n, which are all included in $[-1,2]\times[-1,2]$. Thus $N\cdot\ell(A_n)\leq 9$, so $N \leq 9\lambda^n/4\varepsilon^2$. This implies that

$$\mathrm{cov}(n,2\varepsilon,A) \leq 9\lambda^n/4\varepsilon^2,$$

and taking logs and lims gives us $h_{top}(A)\leq \log\lambda$.

QED ∎

2.5.19 Proposition Let B be a $b\times b$ matrix with integer elements, s.t. $|\det B| = 1$ and let $\alpha_1,\dots,\alpha_b$ be its eigenvalues, with

$\to \neq 0$ en vier $= 1$

$$|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_m| > 1 > |\alpha_{m+1}| \geq |\alpha_b| > 0.$$

Then $B: T^b \to T^b$ is a homeomorphism and

$$h_{top}(B) = \sum_{i=1}^{m} \log|\alpha_i| = \sum_{i=m+1}^{b} \log|\alpha_i^{-1}|$$

Proof: omitted.

2.5.20 Example The topological entropy of the solenoid S (recall example 1.3.10) is $\log 2$.

Proof: Recall that S is topologically conjugate to Φ from Example 2.3.8, so by Corollary 2.5.13 it suffices to study the map $\alpha: \underline{\Phi} \to \underline{\Phi}$, defined by

$$\alpha(\phi) = (2\phi_0 \bmod 1, \phi_0, \phi_1, \dots).$$

Let $d(\phi, \phi') = \sum_{i=0}^{\infty} \frac{|\phi_i - \phi_i'|}{2^i}$, where $|\cdot|$ is the metric on $S^1 = [0,1)/_{0\sim 1}$.

This metric generates the product topology on $(S^1)^{\mathbb{N}}$ (E).

Let $\pi: \underline{\Phi} \to S^1$, be $\pi(\phi) = \phi_0$. Then

$$\pi \circ \alpha(\phi) = 2\phi_0 \bmod 1 \overset{\text{def } E_2}{=} E_2 \circ \pi(\phi),$$

so $\pi: (\underline{\Phi}, \alpha) \to (S^1, E_2)$ is a semiconjugacy. Therefore

$$h_{top}(\alpha) \overset{(E)}{\geq} h_{top}(E_2) \overset{(E)}{=} \log 2.$$

Let $\varepsilon > 0$ and choose $b \in \mathbb{N}$ large enough, s.t. $2^{-b} < \frac{\varepsilon}{2}$.

Then for all $n \in \mathbb{N}$ define

$$A_n \overset{\subset}{=} \underline{\Phi} \left\{ \psi^j \mid (\psi^j)_i = j\, 2^{-(n+b+i)} \bmod 1, j = 0, 1, \dots, 2^{n+2b} - 1 \right\}.$$

It is a simple exercise to show that $A_n \subset \underline{\Phi}$. Next we show that A_n is (n, ε)-spanning. Let $\phi \in \underline{\Phi}$ and

choose $j \in \{0, \dots, 2^{n+2B} - 1\}$ s.t.

$$|\phi_B - j\, 2^{-(n+2B)}| \leq 2^{-(n+2B+1)}.$$

Then $|\phi_i - (\psi^j)_i| \leq 2^{B-i}\, |\phi_B - (\psi^j)_B| \leq 2^{B-i}\, 2^{-(n+2B+1)}$, for all $0 \leq i \leq B$. Then for all $0 \leq m \leq n$, we have

$$d(\alpha^m(\phi), \alpha^m(\psi^j))$$

$$\leq \sum_{i=0}^{B} \frac{2^{m+1}\,|\phi_i - (\psi^j)_i|}{2^i} + \sum_{i=B+1}^{\infty} \frac{1}{2^i}$$

$$\left(\sum_{i=0}^{m} 2^i = 2^{m+1} - 1 \leq 2^{m+1}\right)$$

$$\leq 2^{m+1} \sum_{i=0}^{B} \frac{2^{B-i}\, 2^{-(n+2B+1)}}{2^i} + 2^{-B} \leq \frac{3}{2^B} < \varepsilon, \quad \forall m \Rightarrow \max_{0 \leq m \leq n} d(\cdot,\cdot) \leq \varepsilon$$

so A_n is (n, ε)-spanning. Therefore

$$\mathrm{span}(n, \varepsilon, \alpha) \leq \#A_n = 2^{n+2B}.$$

Taking logs and lims gives $h_{top}(\alpha) \leq \log 2$. QED

3 Symbolic Dynamics

Let $T: X \to X$ be a map and let $\mathcal{P} = \{P_0, \dots, P_{m-1}\}$ be a partition of X. That is $\mathcal{P}$ covers X and $P_i \cap P_j = \emptyset$, when $i \neq j$. Then to each $x \in X$ we may attach an itinerary $(\psi_i(x))_{i \in \mathbb{N}}$ by setting that $\psi_i(x)$ is the unique $j \in \{0, \dots, m-1\}$, s.t. $T^i(x) \in P_j$. In this way, we define a map $\psi: X \to \Sigma_m^+$. If T is invertible, we may also study the passed

itinerary and we obtain a map $\varphi : X \to \Sigma_m$. Now

$$\sigma \circ \varphi(x) = \varphi \circ T(x) \qquad \text{(klinkt logisch)}$$

and so $\varphi : (X, T) \to (\varphi(X), \sigma)$ is a semiconjugacy (which is not necessarily continuous!. The index $\varphi_i(x)$ is called a symbol and we talk about symbolic dynamics.

3.1 Subshifts and codes

3.1.1 Definition A subshift is a closed set $X \subset \Sigma_m$ which is completely invariant under σ, that is, $\sigma(X) \subset X$ and $\sigma^{-1}(X) \subset X$ (which means $\sigma(X) = X$). The whole space is called the full m-shift.

3.1.2 Example Let $X \subset \Sigma_3$, where $X = \{w \in \Sigma_3 \mid w_i \neq 2 \; \forall i\}$. Then X is a subshift. Also the set

$$Y = \{w \in \Sigma_3 \mid w_i w_{i+1} \neq 11 \; \forall i\}$$ (geen twee opeenvolgende 1)

is a subshift.

3.1.3 Definition Let $X_1 \subset \Sigma_{m_1}$ and $X_2 \subset \Sigma_{m_2}$ be subshifts. A code $c : X_1 \to X_2$ is a continuous map which commutes with the shifts, that is $\sigma_2 \circ c = c \circ \sigma_1$.

An injective code is called an embedding.

3.1.4 Remark A surjective code is a topological semiconj. and a bijective code is a topological conjugacy.

3.1.5 Proposition Let $X \subset \Sigma_m$ be a subshift and set

$$W_n(X) := \{ v \in I^n \mid \exists w \in X, \exists i \in \mathbb{Z} \text{ s.t. } w_i \dots w_{i+n-1} = v \},$$

that is, $W_n(X)$ is the set of words of length n appearing in the elements of X. Then

$$h_{top}(\sigma|_X) = \lim_{n\to\infty} \frac{1}{n} \log \# W_n(X).$$

Proof: Since (Σ_m, σ) is expansive, also (X, σ) is expansive. By Thm 2.5.16 (note that 1 is an expansiveness constant) we have that

$$h_{top}(\sigma|_X) = h_{3/4}(\sigma|_X).$$

Observe that $d_n(w, \eta) \geq \frac{3}{4} \iff w_i \neq \eta_i$ for some $i \in \{0, 1, \dots, n\}$. Then $w_0 \dots w_n$ and $\eta_0 \dots \eta_n \in W_{n+1}(X)$ are different words of length $n+1$. Since $\operatorname{diam}_n C_{w_0 \dots w_n} < \frac{3}{4}$, we have that

$$\operatorname{cov}(n, \tfrac{3}{4}, \sigma) \leq \# W_{n+1}(X).$$

on the other hand, if $v \in W_{n+1}(X)$, then there are $i \in \mathbb{Z}$ and $w \in X$ s.t. $w_i w_{i+1} \cdots w_{i+n} = v$. Since $\sigma(X) = X$, we have that $\sigma^i(w) \in X$. Thus the set

$$A = \{ \sigma^{i(v)}(w(v)) \mid v \in W_{n+1}(X) \}$$

is $(n, \frac{3}{4})$-separated? Therefore

$$\mathrm{Cov}(n, \tfrac{3}{4}, \sigma) \geq \mathrm{sep}(n, \tfrac{3}{4}, \sigma) \geq \# W_{n+1}(X)$$

QED

3.1.6 Definition Let $X \subset \Sigma_m$ be a subshift and $b, l \in \mathbb{N}$. Set $n = b + l + 1$ and let $\alpha : W_n(X) \to A_{m'}$, where $A_{m'}$ is a set of m' symbols. Enumerate the elements of $A_{m'}$ by $0, 1, \cdots, m'-1$. A map $c_\alpha : X \to \Sigma_{m'}$ is a (b, l)-block code, if

$$c_\alpha(x)_i = \alpha(x_{i-b}, \cdots, x_i, \cdots, x_{i+l})$$

$\forall i \in \mathbb{Z}$.

3.1.7 Remark Clearly, c_α is continuous and

$$\sigma \circ c_\alpha(x)_i = c_\alpha(x)_{i+1} = \alpha(x_{i+1-b}, \cdots, x_{i+1+l}) = c_\alpha(\sigma(x))_i$$

$\forall i \in \mathbb{Z}$. Thus a block code is a code.

3.1.8 Theorem Every code $c: X \to Y$ is a block code.

Proof: Let $X \subset \Sigma_m$ and $Y \subset \Sigma_{m'}$ be subshifts. Define $\tilde{c}: X \to \{0,1,\dots,m'-1\}$ by setting $\tilde{c}(x) = c(x)_0$. Since X is compact $\tilde{c}$ is uniformly continuous, so $\exists \delta > 0$ s.t. $\tilde{c}(x) \overset{?}{=} \tilde{c}(y)$ provided $d(x,y) < \delta$. Choose $b \in \mathbb{N}$ s.t. $m^{-b-1} < \delta$. Then $\tilde{c}(x)$ depends only on $x_{-b}, \dots, x_0, \dots, x_b$. ?

Thus we may define a map $\alpha: W_{2b+1}(X) \to \{0,\dots,m'-1\}$ s.t. $\alpha(x_{-b},\dots,x_b) = c(x)_0$. Since c commutes with σ, we have that $c = c_\alpha$. (E) Thus c is a (b,b)-block code. QED

3.1.9 Remark There is a canonical way to construct block codes: Let $b, \ell \in \mathbb{N}$ and let X be a subshift. Set

$$c(x)_i = x_{i-b} \dots x_{i+\ell}$$

$\forall i \in \mathbb{Z}$. Then c is a block code with $A_{m'} = W_{b+\ell+1}(X)$. This kind of code is called a higher block presentation of X. (E)

3.2 Subshifts of finite type

3.2.1 Remarks Let $X \subset \Sigma_m$ be a subshift. Since X

is closed (ody), $X^c = \Sigma_m \setminus X$ is open. Thus there exists a cylinder $C_w \subset X^c$. Since $\sigma(X) = X$, we know that w does not appear in any word ($\sigma(y) \in C_w \ \forall y \in X$). Thus w is a forbidden word. Since X^c is open, it is a countable union of cylinders. Therefore, there exists a countable list of forbidden words which do not appear in any el^t of X and atleast one of them appears in every $y \in X^c$. However, there is a small problem. Namely, if w is a forbidden word, so is wv for any word v. How to find the shortest forbidden words? Solution: order the cylinders "centered" at 0 s.t. first comes cylinder C_{y_0}, then cylinder $C_{y_0 y_1}$, then $C_{y_{-1} y_0 y_1}$ and so on. Every cylinder is a finite union of these cylinders. Then

$$X^c = \bigcup_{C_w \subset X^c} C_w =: \bigcup_{i=1}^{\infty} C_i .$$

Let C_{w_1} be the smallest cylinder in this order included in X^c. Remove the cylinder C_{w_1} and all cylinder C_i s.t. $C_i \subset C_{w_1}$. Let $\{C_i'\}_{i=1}^{\infty}$ be the collection obtained this way. Now let $Y_1 = X^c \setminus \bigcup_{b \in \mathbb{Z}} \sigma^b(C_{w_1})$. Let C_{w_2} be the smallest cylinder C_{i_1}' s.t. $C_{w_2} \cap Y_1 \neq \emptyset$. Remove C_w from the collection $\{C_i'\}_{i=1}^{\infty}$, for all words w, for which

$C_w \subset C_{w_2}$ and let $Y_2 = Y_1 \setminus \bigcup_{b\in\mathbb{Z}} \sigma^b(C_{w_2})$. Continuing this process, we obtain a list of words w_i, which are called forbidden words as a substring, that is, there are $i \in \mathbb{Z}$, $j \in \mathbb{N}$ s.t.

$$y_i \cdots y_{i+b_j-1} = w_j,$$

for some forbidden word w_j of length b_j. Clearly, if w_i and w_j are forbidden words, then $C_{w_j} \not\subset C_{w_i}$ and $C_{w_i} \not\subset C_{w_j}$, if $i \neq j$.

3.2.2 Definition A *subshift of finite type* is a subshift $X \subset \Sigma_m$, where the associated list of forbidden words is finite. If the longest forbidden word has length $b+1$, we say that X is a *b-step subshift (of finite type)*. A 1-step subshift is called a *topological Markov chain*.

3.2.3 Example Let A be a $m \times m$ matrix whose el$^{\underline{s}}$ are 0's or 1's and whose rows and columns are indexed with $0, \dots, m-1$. Let Γ_A be the directed graph

determined by A, that is the vertex set of Γ_A is $\{0, \dots, m-1\}$, and there is a directed edge from i to $j \iff A_{i,j} = 1$. Let

$$\Sigma_A := \{x \in \Sigma_m \mid A_{x_i x_{i+1}} = 1 \ \forall i \in \mathbb{Z}\}.$$

It is easy to see that $\sigma(\Sigma_A) = \Sigma_A$ (logisch), and that Σ_A is closed. Thus Σ_A is a subshift. Every elt of Σ_A can be thought of as a two sided infinite walk on Γ_A, where one walks along the directed edges of Γ_A and at time i one is at the vertex x_i. The matrix A is called the adjacency matrix of Σ_A. We make the following simple observation about the forbidden words of Σ_A: If $x \in \Sigma_m \setminus \Sigma_A$, then there is $i \in \mathbb{Z}$ s.t. $A_{x_i x_{i+1}} = 0$. Then either $x_i x_{i+1}$ is a forbidden word or one of x_i and x_{i+1} is. Thus

$$\{x \in \Sigma_m \mid x \text{ is a forbidden word of } \Sigma_A\} \subset \{0, \dots, m-1\} \cup \{ij \mid A_{ij} = 0\}.$$

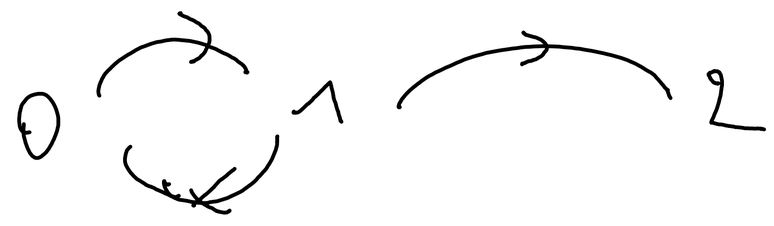
(longest forb. word $= 2$; $\leq m + m \cdot m$)

In particular, the set of forbidden words is finite, so Σ_A is a subshift of finite type and more specifically a topological Markov chain. For example, let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

then

Γ^A:



Then the word 2 will not appear in any $x \in \Sigma_A$, since we cannot continue the walk from the vertex 2, so 2 is a forbidden word. $\{2, 00, 11\}$ is the list of forbidden words.

3.2.4 Definition A subshift Σ_A^v determined by an adjacency matrix A is called a vertex shift.

3.2.5 Remark To any directed graph, one may attach an adjacency matrix by setting $A_{ij} = 1$ $\Leftrightarrow$ there is a directed edge from vertex i to vertex j.

3.2.6 Definition Let Γ be a finite directed graph, where between vertices i and j, there is a finite number of directed edges. To Γ one may attach an adjacency matrix B, where B_{ij} is given by the number of directed edges from the vertex i to j. This kind of an adjacency matrix B determines a edge shift Σ_B^e as the set of two sided infinite walks on edges of Γ, where the edge e may follow the edge e', if the terminal

vertex of e' is the initial vertex of e.

3.2.7 Remark: An edge shift is a subshift of finite type (E).
If A is an adjacency matrix with 0's and 1's, then the map $c_\infty : \Sigma_A^V \to \Sigma_A^E$ where $\alpha((i,j)) = e(i,j)$ is a bijective $(0,1)$-block code, so it's a topological conjugacy (E) Conversly, there is a canonical conjugacy From edge shift to some vertex shift (E)

↳ bit more complicated then $\Sigma_A^V \to \Sigma_A^E$

3.2.8 Thm A subshift of finite type is conjugate to a vertex shift

proof Let X be a b-step subshift with $b > 0$ ($b=0$ is trivial) we have to construct a graph. Trick: Let Γ be a graph whose vertices are the el^{ts} of $W_b(X)$ We define edges of Γ by saying that there is an ↳ finite?
edge from vertex $x_1 \dots x_b$ to vertex $x'_1 \dots x'_b$ iff
↳ word

$$x_1 \dots x_b x'_b = x_1 x'_1 \dots x'_b \in W_{b+1}(X)$$

Let A be the adjacency matrix related to Γ.
Define a code $c : X \to \Sigma_A^V$ by

$$c(x)_i = x_i x_{i+1} \dots x_{i+b-1}$$

↳ length b word ?

Then c is well defined ($\in$ is in Σ_A^V ?), it's continuous and $c \circ \sigma = \sigma \circ c$ (E). Clearly, c is injective, since if $x, y \in X$ s.t. $x \neq y$. There is i s.t. $x_i \neq y_i$ so $c(x) \neq c(y)$. To prove surjectivity, take $y \in \Sigma_A^V$

Let $x \in \Sigma_M$ be s.t., $x_i = (\overbrace{Y_i}^{\text{a word of length } b})_1$, for all $i \in \mathbb{Z}$. We have to show that $x \in X$, since $y \in \Sigma_A^{\nu}$, we have that $(y_{j+p})_1 = (Y_j)_{1+p}$ for all $j \in \mathbb{Z}$ (*)

Recall: $z_1 \cdots z_b z'_b = z_1 z'_1 \cdots z'_b$ (think about it yourself)

an $p \in \{1, \ldots, b-1\}$ in particular,

$$x_i x_{i+1} \cdots x_{i+b-1} = y_i \in W_b(X)$$

Thus there are no forbidden words of length $\ell < b$ in X.

Now check length $b+1$. Since $y \in \Sigma_A^{\nu}$, we have $A_{y_i y_{i+1}} = 1$

that is $x_i \cdots x_{i+b} = x_i \cdots x_{i+b-1} (y_{\underbrace{i+b}_{= i+1+b-1}})_1$ (def of x)

$\overset{(*)}{=} x_i \cdots x_{i+b-1} (Y_{i+1})_b = y_i (y_{i+1})_b \in W_{b+1}(X)$. Thus

there are no forbidden words of length $b+1$. Since X is a b-step subshift, we have $x \in X$. Clearly, $\sigma(x) =$

QED

3.2.9 corollary A subshift of finite type is conjugate to an edge shift

3.2.10 Remark: By Theorem 3.2.8 every subshift of finite type is conjugate to a topological markov chain. Thus after conjugacy, the past is independent of the future, that is, if $\ldots x_{-2}x_{-1}x_0$ is an allowed sequence and $x_0 x_1 x_2 \ldots$ is an allowed sequence, then $\ldots x_{-2} x_{-1} x_0 x_1 x_2 \ldots$ is an allowed sequence (gluing).

3.2.11 example: Let Σ_A^ν be a vertex shift determined by an adjacency matrix A. Then the number of n-periodic points is $\mathrm{tr}(A^n)$

proof: We show first that $(A^n)_{ij}$ is the number of n-step walks from vertex i to vertex j. This is clear if $n = 1$. Now $(A^n)_{ij} = \sum_{b=0}^{m-1} (A^{n-1})_{ib} A_{bj}$ (how you multiply matrices). By induction hypothesis, $(A^{n-1})_{ib}$ is the number of $(n-1)$-step walks from i to b and one may step from b to j iff $A_{bj} = 1$. Thus, $(A^n)_{ij}$ is the number of n-step walks from i to j. In particular $(A^n)_{ii}$ is the number of n-step walks from i to i. This can be repeated and $x = \ldots i j_1 \ldots j_{n+1} i \in \Sigma_A^\nu$ clearly, x is n-periodic. Conversely, every n-periodic point defines some n-step walk from i to i. Thus, the number of n-periodic points is

$$\sum (A^n)_{ii} = \mathrm{tr}(A^n)$$

QED

A is $m \times m$

3.3 Perron - Frobenius theorem

3.3.1. Def A matrix A is positive (non-negative)

if all of its el^{ts} are positive (non-negative). A non-negat. (↳ you may have zero)
square matrix A is reducible if $\forall i,j, \exists n$ s.t. $(A^n)_{ij} > 0$ (↳ there is a walk $i \to j$)
otherwise A is reducible. A non-negative square matrix
A is primitive, if there exists n s.t. A^n is positive. (↳ means no walk $i \not\to j$)
(so prim $\Rightarrow$ irred) A non-negative square matrix is
stochastic if $\sum_{i=0}^{m-1} A_{ij} = 1 \quad \forall i$.

3.3.2 Remark A non-negative square matrix A with integer el^{ts} is primitive iff, for some n, A has an n-step from i to j for all i,j. Irreducibility means that there is a walk from i to j.
A non-negative square matrix is stochastic iff $(1,1,..,1)$ is an eigenvector of A with eigenvalue 1.

3.3.3 Lemma Let $L: \mathbb{R}^b \to \mathbb{R}^b$ be a linear map. Assume that there exists a compact set P s.t. $0 \in P$ and $L^i(P) \subset \text{Int}(P)$ for some i. Then $|\sigma| < 1$ for all eigenvalues σ of L. Here eigenvalue refers (↳ eigenval)
to the roots of the characteristic polynomial.

Vanaf hier ik niet gelopt

Last part of proof of thm 3.3.4

Set $C = \{x \in \mathbb{R}^m \mid x_j \geq 0 \quad \forall j = 1, \dots, m\}$

Since A is non negative, we have that $A(C) \subset C$.

Assume that $v \in C \setminus \{0\}$ is an eigenvector with eigenvalue σ. By (1) we have $0 \leq \sigma < d$, since $A(C) \subset C$.

3.3.5 Corollary Let A be a primitive stochastic matrix. Then 1 is a simple root of the characteristic polynomial, for A and A^T have a positive eigenvector corresponding to the eigenvalue 1 and all other eigenvalues of A and A^T satisfy $|\sigma| < 1$ and corresponding eigenvectors are not non-negative

3.3.6 Theorem (Frobenius) Let A be a non-negative irreducible square matrix. Then A has an eigenvalue λ with the following properties:

as person
a) $\lambda > 0$,
b) λ is a simple root of a characteristic polynomial,
c) there is a positive eigenvector corresponding the eigenvector λ
d) if μ is another eigenvector, then $|\mu| \leq \lambda$
e) if A has b eigenvalues with $|\mu| = \lambda$, then the spectrum of A is invariant under rotation by $\frac{2\pi}{b}$.

proof: omitted QED

3.3.7 Theorem Let A be an $m \times m$ matrix with $a_{ij} \in \{0,1\}$. Let Σ_A^ν be the corresponding vertex shift: Then $h_{top}(\sigma_A) = \log \lambda$ where λ is the eigenvalue of A with largest absolute value.

proof: Set $S_n = \sum_{i,j=1}^{m} (A^n)_{ij}$ Trick. We show first that $\lim_{n\to\infty} \frac{\log S_n}{n} = \log \lambda$

Observe that (Ξ) Look very difficult

$$\frac{1}{m} \|A^n\| \leq S^n \leq m^2 \|A^n\|.$$

Recalling Gelfand's formula, which states that

$$S(A) := \max\{|\mu| \mid \mu \text{ is an eigenvector of } A\}$$
$$= \lim_{n\to\infty} \|A^n\|^{1/n} \quad (\Xi)$$

We observe that

$$\lim_{n\to\infty} \frac{\log S_n}{n} = \lim_{n\to\infty} \frac{\log \|A^n\|}{n} \overset{\text{log rules}}{=} \lim_{n\to\infty} \log \|A^n\|^{1/n} = \log|\lambda|$$

how is this related to entropy? By proposition 3.1.5

$$h_{top}(\sigma_A) = \lim_{n\to\infty} \frac{1}{n} \log \#W_n(\Sigma_A^V) \overset{E}{=} \lim_{n\to\infty} \frac{1}{n} \log S_n = \log(\lambda)$$

QED ∎

3.3.8 Remark The claim of Thm 3.3.7 is valid also for the edge shift.

3.4 Shift equivalence

3.4.1 Definition Non-negative integer square matrices A and B are elementary strong if there exists non-negative integer matrices U and V s.t. $A = UV$ and $B = VU$. Matrices A and B are strong shift equivalent if there exists some $n \in \mathbb{N}$ and non-negative integer square matrices $C_1, \dots, C_n$ s.t. $C_1 = A$, $C_n = B$ and C_i and C_{i+1} are elementary strong shift equivalent $\forall\, i = 1, \dots, n-1$.

3.4.2 Example The matrices

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \quad \text{and} \quad B = (3)$$

are strong shift equivalent but not elementary strong shift eq

proof: Strong shift equivalence (E). Suppose that $A=UV$ and $B=VU$ for some non-negative integer matrices U and V. Then $V=(a\ b\ c)$ and $U=\begin{pmatrix} d \\ e \\ f \end{pmatrix}$. Thus $A=UV=\begin{pmatrix} da & db & dc \\ ea & eb & ec \\ fa & fb & fc \end{pmatrix}$

Since $da=1$ then $d\neq 0$. Since $dc=0 \Rightarrow c=0$ } $fc=1$

QED

3.4.3 Thm (Williams) Two edge shifts Σ_A^e and Σ_B^e are topologically conjugate iff A and B are strong shift equivalent

proof: $\Rightarrow$ Difficult so omitted

$\Leftarrow$ It is enough to consider the case that A and B are elementary strong shift equivalent (by the transitivity of topological conjugation)

Let $A=UV$ and $B=VU$ and let Γ_A and Γ_B be disjoint graphs corresponding to A and B. Let A be a $b\times b$-matrix and B a $\ell\times\ell$-matrix. Then U is a $b\times\ell$-matrix and V is an $\ell\times b$-matrix. We interpret U_{ij} as the number of new edges from vertex i of Γ_A to vertex j of Γ_B and similarly for V_{ji}. Since $A_{pq}=\sum_{j=1}^{\ell} U_{pj}V_{jq}$, the number of edges in Γ_A from p to q is the number of 2-step walks from p to q via Γ_B. Thus we may define a bijection ϕ from the set of edges of Γ_A to pairs of edges UV, that is, $\phi(a)=UV$, where U is an edge connecting the initial vertex of a to some vertex of Γ_B and v is an edge connecting this vertex back to the terminal vertex of a. Similarly, there is a bijection ψ from the edges of Γ_B to pairs of edges VU.

Let $\ldots a_{-1}a_0a_1\ldots \in \Sigma_A^e$. Then applying ϕ to every symbo

we get $\cdots \phi(a_{-1})\,\phi(a_0)\,\phi(a_1)\cdots = \cdots U_{-1}V_{-1}U_0V_0U_1V_1\cdots$ (4)
Since $t(v_i) = i(v_{i+1})$, where t denotes the terminal vertex and i the initial one, we may apply ψ^{-1} to define an edge $b_i = \psi^{-1}(v_i v_{i+1})$ in Γ_B. Applying ψ^{-1} to (4), we obtain

$$\cdots b_{-1}b_0b_1\cdots \in \Sigma_B^e$$

if $a \neq \tilde{a} \in \Sigma_A^e$, then $a_i \neq \tilde{a}_i$ for some i, so $u_iv_i \neq \tilde{u}_i\tilde{v}_i$ and then either $u_i \neq \tilde{u}_i$ or $v_i \neq \tilde{v}_i$. Thus $b_{i-1} \neq \tilde{b}_{i-1}$ or $b_i \neq \tilde{b}_i$. Thus this map is an injection. Reversing the roles of A and B we find an injection in the other direction. Which are inverses of each other (E). Clearly σ commutes with this map.

QED ∎

3.4.4 Definition Non-negative integer square matrices A and B are shift equivalent if there exist non-negative integer matrices U & V and $\ell \in \mathbb{N}$ s.t.

$$A^\ell = UV,\ B^\ell = VU,\ AU = UB \text{ and } BV = VA.$$

3.4.5 Remark Williams (1973) conjectured that irreducible shift equivalent matrices are strong shift equivalent but Kim and Roush (1999) constructed a counter example.

3.5 Substitutions

3.5.1 Definition Let $\mathcal{A}_m := \{0, 1, \ldots, m-1\}$ and $\mathcal{A}_m^* = \bigcup_{i=1}^{\infty} \mathcal{A}_m^i$. A substitution is a map

$$s: \mathcal{A}_m \to \mathcal{A}_m^*.$$

A substitution induces a map $s: \mathcal{A}_m^* \to \mathcal{A}_m^*$ and $s: \Sigma_m^+ \to \Sigma_m^+$ by the formula $s(a_0a_1\ldots) := s(a_0)s(a_1)\cdots$

3.5.2 Remark Denote by $|w|$ the length of $w \in A_m^*$.

We will assume that $|s(a)| > 1$ for some $a \in A_m$ and $|s^k(b)| \xrightarrow{k\to\infty} \infty$ for all $b \in A_m$. Note that the maps are continuous.

3.5.3 Example Let $A_2 = \{0,1\}$ and define

$$s: A_2 \to A_2^* \text{ by } s(0) = 01 \text{ and } s(1) = 10.$$

Then $s^2(0) = s(01) = 0110$, $s^3(0) = 01101001$,

$s^4(0) = 0110100110010110, \ldots$

For a fixed $w \in A_2^*$, let $\bar{w}$ denote the word where the roles of 0 and 1 are reversed. We show

$$s^{n+1}(0) = s^n(0)\overline{s^n(0)}.$$

When $n = 1$, the claim is true. Suppose that it's true for n.

Then

$$s(s^n(0)) = s^{n+1}(0) = s(s^n(0)) \overset{IH}{=} s(s^{n-1}(0)\overline{s^{n-1}(0)})$$

$$= s^n(0)\overline{s(s^{n-1}(0))} = s^n(0)\overline{s^n(0)}$$

since $s(\bar{0}) = s(1) = 10 = \overline{01} = \overline{s(0)}$ and

$s(\bar{1}) = s(0) = 01 = \overline{10} = \overline{s(1)}$.

$s^{n(i)}(0)_i = s^{n(i)+1}(0)_i = \ldots$

Therefore, for all i, there is $n(i)$ s.t. $s^n(0)_i$ is constant for all $n \geq n(i)$. Thus $s^n(0)$ converges to an infinite sequence

$$M = 0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\ldots$$

This sequence is called (Prouhet-Thue-) Morse sequence.

Since s is continuous, we have

$$s(M) = s\left(\lim_{n\to\infty} s^n(0)\right) = \lim_{n\to\infty} s^{n+1}(0) = M,$$

that is, M is a fixed point of s. Similarly, one sees that $s(\bar{M}) = \bar{M}$. These are the only fixed points of s. (B)

3.5.4 Proposition Every substitution s has a periodic point in Σ_m^+.

Proof: consider the map $a \mapsto s(a)_0$. Since $\mathcal{A}_m$ has m elts, a simple pigeonhole argument implies that there is some a in $\mathcal{A}_m$ and some $n \in \{0,\ldots,m\}$ s.t. $s^n(a)_0 = a$. (E) If $|s^n(a)| = 1$, then $s^n(a) = a$ and $aaa\ldots \in \Sigma_m^+$ is a fixed point of s^n and, thus, an n-periodic point for s. In the other case, we have that $|s^n(a)| \xrightarrow{n\to\infty} \infty$ and the sequence $\left(s^{in}(a)\right)_{i=1}^{\infty}$ converges to a point $x \in \Sigma_m^+$ which is a fixed point of s^n. (E)

QED ∎

3.5.5 Remark If $s(a)_0 = a$ and $|s(a)| > 1$, then

$$s^\infty(a) := \lim_{n\to\infty} s^n(a)$$

exists ($\in$) and it is a fixed point of s.

If $x \in \Sigma_m^+$ is a fixed point of s and $|s(x_0)| > 1$, then $x = x_0 x_1 x_2 \dots$

$s^n(x_0) \xrightarrow{n\to\infty} x$. If $|s(a)| > 1$ for all $a \in \mathcal{A}_m$, then s has at most m fixed points. If $s(a) = a$ and $s(b) = b$ for some $a \neq b$, then there are uncountably many fixed points. Observe that

$$\Sigma_s(a) := \underline{\mathcal{O}_\sigma^+(s^\infty(a))}$$

is a subshift.

3.5.6 Definition A substitution $s: \mathcal{A}_m \to \mathcal{A}_m^*$ is irreducible

(versteh n) { if for all $a, b \in \mathcal{A}_m$, there exists $n \in \mathbb{N}$ s.t. $s^n(a)$ contains b.

A substitution s is primitive if there exists $n \in \mathbb{N}$ s.t.

(es gibt n für alles) { $s^n(a)$ contains b for all $a, b \in \mathcal{A}_m$

3.5.7 Theorem Assume that s is an irreducible substitution on $\mathcal{A}_m$. If $s(a)_0 = a$ for some $a \in \mathcal{A}_m$, then s is primitive and the subshift $(\Sigma_s(a), \sigma)$ is minimal.

Proof: Since $s(a)_0 = a$ we have $s^n(a)_0 = a$ for all $n \in \mathbb{N}$.

Since s is irreducible, for all $b \in \mathcal{A}_m$, there exists $n(b)$ s.t.

$s^{n(b)}(a)$ contains b. Thus for all $n \geq n(b)$, we have

$$s^n(a) = s^{n(b)}(s^{n-n(b)}(a)) \text{ and}$$

$$s^{n-n(b)}(a)_0 \overset{\text{assumption}}{=} a.$$

Hence, $s^n(a)$ contains b for all $n \geq n(b)$. Thus, $s^n(a)$ contains all symbols $b \in \mathcal{A}_m$ for all $n \geq N := \max_{b \in \mathcal{A}_m} n(b)$. Since s is irreducible, for all $b \in \mathcal{A}_m$, there is $b(b) \in \mathbb{N}$ s.t. $s^{b(b)}(b)$ contains a. Since $s(a)_0 = a$, we have

$$s^n(b) = s^{n-b(b)}(s^{b(b)}(b))$$

contains a for all $n \geq b(b)$. Therefore, for all $b \in \mathcal{A}_m$, the word $s^n(b)$ contains all symbols of $\mathcal{A}_m$ when $n \geq \max_{c \in \mathcal{A}_m} b(c) + N$. Thus s is primitive. By Remark 3.5.5 and irreducibility of s, we have that $s^\infty(a)$ exists.

Now for the minimality. By Theorem 2.1.10 it is enough to prove that $s^\infty(a)$ is almost periodic, that is, for all n, the word $s^n(a)$ appears in $s^\infty(a)$ infinitely often with bounded gaps. This follows from the fact that a appears in $s^\infty(a)$ infinitely times with bounded gaps (E). Since we may apply s^n to these a's and the distance between $s^n(a)$'s depends only on n and the distance between these a's. QED

3.5.8 Definition Let $u, v \in A_m^*$. Let $N_v(u)$ be the number of u's in v. The composition matrix $M(s)$ of s is the $m \times m$ matrix s.t.

$$M(s)_{ij} = N_i(s(j)) \qquad \text{(non-neg integer matrix)}$$

3.5.9 Proposition a) We have $M(s^n) = M(s)^n$

b) The matrix $M(s)$ is irreducible / primitive $\Leftrightarrow$ s is irreducible / primitive

c) For all $w \in A_m^*$ the numbers $N_i(w)$ define a vector $N(w) \in \mathbb{R}^m$. Then $N(s(w)) = M(s) N(w)$

d) If $|s(i)| = \ell \ \forall i \in A_m$ (so all same length), then

$$\sum_{i \in A_m} M(s)_{ij} = \ell \qquad \forall j \in A_m \quad \text{and}$$

the matrix $(\ell^{-1} M(s))^T$ is stochastic.

Proof: a) If $n = 1$, the claim is trivial. Assume that it is true for n. Then

first take power ∇ (then ij'th el.)

$$M(s)^{n+1}_{ij} \overset{\text{matrix mult def}}{=} \sum_b M(s)^n_{ib} M(s)_{bj} \overset{IH}{=} \sum_b M(s^n)_{ib} M(s)_{bj}$$

$$\overset{\text{def } M}{=} \sum_b N_i(s^n(b)) N_b(s(j)).$$

$\parallel$ written $N_i(s^{n+1}(j))$

Now i appears in $s^{n+1}(j)$ $\Leftrightarrow$ there exists b s.t. b appears in $s(j)$ and i appears in $s^n(b)$. Thus

$$M(s^{n+1})_{ij} = N_i(s^{n+1}(j)) = \sum_b N_i(s^n(b))\, N_b(s(j)) \overset{1}{=} M(s)^{n+1}_{ij}$$

(a) – d) (E)

QED ∎

3.5.10 Theorem Let $s : A_m \to A_m^*$ be a primitive substitution and let λ be the largest eigenvalue of $M(s)$.

↳ we will prove that λ is real

Then for all $a \in A_m$, we have

a) $\lim_{n\to\infty} \lambda^{-n} N(s^n(a))$ is an eigenvector of $M(s)$ corresponding to λ

↳ so limit exists

b) $\lim_{n\to\infty} \frac{|s^{n+1}(a)|}{|s^n(a)|} = \lambda$ (how fast does it increase?)

c) $v := \lim_{n\to\infty} \frac{N(s^n(a))}{|s^n(a)|}$ is an eigenvector of $M(s)$ corresponding to λ and $\sum_{i=0}^{m-1} v_i = 1$

Proof: a) By Proposition 3.5.9, $N(s^n(a)) = M(s)^n N(a)$ and $M(s)$ is primitive. By Theorem 3.3.4, λ is positive, it is a simple root of the characteristic polynomial, it has a positive eigenvector v, $M(s)$ has no other non-negative eigenvectors and $|\sigma| < \lambda$ for other eigenvalues σ of $M(s)$. Write $N(a) = \alpha v + \beta u$, where $\alpha > 0$ and u belongs to the eigenspace corresponding to

↳ spectrum

$sp(M(s)) \setminus \{\lambda\}$. Let $\max_{\sigma \in sp(M(s)) \setminus \{\lambda\}} |\sigma| < \mu < \lambda$. Then there exists $C > 0$ s.t. $|M(s)^n u| \leq C \mu^n |u|$. Therefore

↳ convergence

$$\lim_{n\to\infty} \frac{N(s^n(a))}{\lambda^n} = \lim_{n\to\infty} \frac{M(s^n N(a)}{\lambda^n}$$

$$= \lim_{n\to\infty} \frac{\alpha M(s)^n v + \beta M(s)^n u}{\lambda^n} = \alpha v + \lim_{n\to\infty} \beta C \left(\frac{N}{\lambda}\right)^n = \alpha v \quad (\to 0)$$

b) Since $|s^n(a)| = \sum_i N_i(s^n(a))$ and since by (a) we have

$\lim_{n\to\infty} \lambda^{-n} N_i(s^n(a)) = \alpha v_i$, we obtain

for any small $\varepsilon > 0$ that

$$\lim_{n\to\infty} \frac{|s^{n+1}(a)|}{|s^n(a)|} \leq \lim_{n\to\infty} \frac{\lambda^{n+1}(\sum_i \alpha v_i + \varepsilon)}{\lambda^n(\sum_i \alpha v_i - \varepsilon)} = \frac{\lambda \sum_i \alpha v_i + \varepsilon}{\sum_i \alpha v_i - \varepsilon}$$

Since this is true for all small $\varepsilon > 0$, we have that

$\lim_{n\to\infty} \frac{|s^{n+1}(a)|}{|s^n(a)|} \leq \lambda$ Similarly,

$$\lim_{n\to\infty} \frac{|s^{n+1}(a)|}{|s^n(a)|} \geq \lim_{n\to\infty} \frac{\lambda^{n+1}(\sum_i \alpha v_i - \varepsilon)}{\lambda^n(\sum_i \alpha v_i + \varepsilon)} = \frac{\lambda \sum_i \alpha v_i - \varepsilon}{\sum_i \alpha v_i + \varepsilon}$$

So $\lim_{n\to\infty} \frac{|s^{n+1}(a)|}{|s^n(a)|} = \lambda$

c) (E)

QED ∎

3.5.11 Theorem Let s be a primitive substitution, $x = s^\infty(a)$ be a fixed point of s and let l_n be the number of different words of length n appearing in x. Then there is a constant $C > 0$ s.t. $l_n \leq C \cdot n$ for all $n \in \mathbb{N}$. In particular, the topological entropy of $(\Sigma_s(a), \sigma)$ is 0.

($l_n = \mathcal{O}(n)$; $\Sigma_s(a)$: orbit closure of x)

Proof: Let $\underline{v}_b = \min_{b \in \mathcal{A}_m} |s^b(b)|$ and $\bar{v}^b = \max_{b \in \mathcal{A}_m} |s^b(b)|$. By Remark 3.5.2, we have that

$$\lim_{b \to \infty} \underline{v}_b = \lim_{b \to \infty} \bar{v}^b = \infty$$

monotonically. Thus, for all $n \in \mathbb{N}$, there is $b = b(n)$ s.t.

$$\underline{v}_{b-1} \leq n \leq \underline{v}_b .$$

Therefore, for any word w in x with length n, there are two succesive symbols bc in x s.t. w is included in $s^b(bc)$ ($x = s^b(x) = s^b(x_1) \dots s^b(x_2)$)

We may assume that w starts in $s^b(e)$. Let λ be the largest eigenvalue of $M(s)$. As in the proof of theorem 3.5.10, we see that for all non-negative vectors y there exists constants $C_1(y)$ and $C_2(y)$ s.t.

$$C_1(y)\lambda^l \leq |M(s)^l y| \leq C_2(y)\lambda^l \quad \forall l \in \mathbb{N}.$$

Since for any $a \in \mathcal{A}_m$, we have

$$|s^l(a)| = \sum_i N_i(s^l(a)) = \sum_i (M(s)^l N(a))_i$$

There are constants C_1 and C_2 s.t.

$$C_1 \lambda^l \leq \underline{v}_l \leq \bar{v}^l \leq C_2 \lambda^l$$

for all $l \in \mathbb{N}$. Since for all $b \in \mathcal{A}_m$ the word $s^b(bc)$ contains at most $\bar{v}^b$ words of length n starting in $s^b(b)$, we have

$$h_n \leq m^2 \bar{v}^b \leq c_2 m^2 d^b = \frac{c_2}{c_1} m^2 d c_1 d^{b-1} \leq \frac{c_2}{c_2} m^2 d v_{b-1}$$

$$\leq \frac{c_2}{c_1} m^2 d n = C n.$$

The entropy claim follows from proposition 3.1.5 $\therefore C$

QED ∎

↳ Hclucw for finite

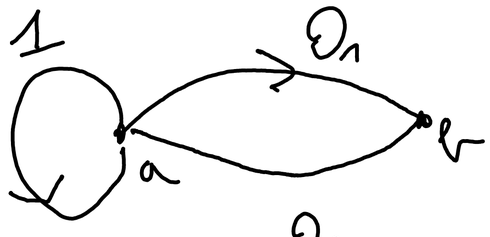
3.6 Sofic Shifts

3.6.1 Definition A subshift $X \subset \Sigma_m$ is sofic if it is a factor of a subshift of finite type.

3.6.2 Remark By Corollary 3.2.9 a subshift X is sofic iff there exist an adjacency matrix A and a surjective code $\epsilon : \Sigma_A^e \to X$.

Since the identity map is a conjugacy, every subshift of finite type is sofic.

3.6.3 Example (Even system of weiss)

Γ: 

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

The A is the adjacency matrix of Γ. Let $X \subset \Sigma_2$ be the set of finite sequences s.t. between each pair of 1's

there are an even number of 0's. Then X is a subshift.

Define a map $c: \Sigma_A^e \to X$, where c is a block code determined by α, where $\alpha(1)=1$, $\alpha(0_1)=\alpha(0_2)=0$. Then c is a surjective code. Thus, X is sofic but it is not a subshift of finite type (E).

↳ done in previous exercise

3.6.4 Definition Let Γ be a finite directed graph whose edges are labeled with el^{ts} of A_m in such a way that the same label may appear several times. Let $X_\Gamma \subset \Sigma_m$ be the set generated by all two sided infinite walks on Γ. If (X,σ) is conjugate to (X_Γ,σ), then Γ is a representation of X. (for example: graph in exa - 3.6.3

→ you can give more edges the same label

3.6.5 Remark a) The set X_Γ is clearly closed and shift invariant, so it is a subshift.

b) Renaming $0_1=0_2=0$ we have that the graph from example 3.6.6 is a representation of the even system of weiss.

3.6.6 Theorem A subshift $X \subset \Sigma_m$ is sofic $\iff$

it has a representation Γ.

Proof: $\Rightarrow$ Let X be sofic. Then there exists an adjacency matrix A and a surjective code $c: \Sigma_A^e \to X$. By theorem 3.1.8, c is a block code. By changing to a higher block presentation (see Remark 3.1.9), we may assume that c is a $(0,0)$-block code. Thus, there exists $\mathcal{A}_{m_1} = \{$edges of the higher block presentation$\}$ and a map $\alpha : \mathcal{A}_{m_1} \to \mathcal{A}_m$ for which $c(x)_i = \alpha(x_i)\ \forall i$. Let Γ be the graph corresponding to this higher block presentation. Then label the edges of Γ by $\mathcal{A}_m$ using α, that is, the label of e is $\alpha(\alpha)$. Then Γ is a representation of X.

$\Leftarrow$ Let Γ be a labeled graph and $c_2 : X_\Gamma \to X$ be a conjugacy. Give new names to the edges of Γ s.t. every edge has a different name. Let A be the corresponding adjacency matrix. Then Σ_A^e is a subshift of finite type. Let $c_1 : \Sigma_A^e \to X_\Gamma$ using the original labels of Γ. Then c_1 is surjective, so X is a factor of Σ_A^e via $c = c_2 \circ c_1$.

QED ∎

3.6.7 Remark Sofic shifts appear naturally in many

different occations, for example geodesic flows and data foraging.

4. Ergodic Theory

↳ study of asymptotic behaviour of typical points under measurable dynamics

4.0.1 Definition A measurable dynamical system is a quadruple $(X, \mathcal{B}, \mu, T)$ where X is a set, $\mathcal{B}$ is a σ-algebra, μ is a measure on $\mathcal{B}$ and $T: X \to X$ is measurable,

Recap: σ-algebra: $\mathcal{B} \subset \mathcal{P}(X)$

$\phi \in \mathcal{B}$

$A \in \mathcal{B} \Rightarrow A^c \in \mathcal{B}$

$A_i \in \mathcal{B} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$

measure: $\mu: \mathcal{B} \to \mathbb{R}_+$

$\mu(\phi) = 0$

$A_i \in \mathcal{B} \quad A_i \cap A_j = \phi$

$\Rightarrow \mu(\cup A_i) = \sum \mu(A_i)$

that is, $T^{-1}(B) \in \mathcal{B} \quad \forall B \in \mathcal{B}$. If $\mu(X) < \infty$, then X (or more precisely $(X, \mathcal{B}, \mu)$) is a finite measure space and if $\mu(X) = 1$, then X is a probability measure. The map T is non-singular if $\mu(T^{-1}(B)) = 0 \quad \forall B \in \mathcal{B}$ with

$\mu(B) = 0$. The map T preserves the measure μ or μ is invariant under T or is T-invariant, if $T_*\mu = \mu$, that is, $T_{*\mu}(B) = \mu(T^{-1}(B)) = \mu(B) \quad \forall B \in \mathcal{B}$.

4.0.2. Remark A measurable flow on a measure space $(X, \mathcal{B}, \mu)$ is a map $T : X \times \mathbb{R} \to X$ where $(x,t) \mapsto T^t(x)$, which is measurable on the σ-algebra $\mathcal{B} \times \mathcal{B}$, where $\mathcal{B}$ is the lebesgue σ-algebra on $\mathbb{R}$. The flow preserves μ if $T^t_{*\mu} = \mu \quad \forall t \in \mathbb{R}$.

4.1 Recurrence

4.1.1. Theorem (Poincaré reccurence theorem)

Let T be a measure preserving map on a finite measure space $(X, \mathcal{B}, \mu)$ if $A \in \mathcal{B}$ then for μ-almost every $x \in A$, there is $n \in \mathbb{N} \setminus \{0\}$ s.t. $T^n(x) \in A$. In particular, for μ-almost all $x \in A$ there are infinitely many $n \in \mathbb{N}$ s.t. $T^n(x) \in A$.

P(x) holds for μ-a.e. x if $\mu(\{x \mid P(x) \text{ does not hold}\}) = 0$

Proof: Let

$$B = \{x \in A \mid T^b(x) \notin A \text{ for all } b \in \mathbb{N} \setminus \{0\}\} = A \setminus \bigcup_{b=1}^{\infty} T^{-b}(A).$$

Then $B \in \mathcal{B}$ and $T^{-b}(B) \in \mathcal{B}$ for all $b \in \mathbb{N}$ and

Lazie def 4.0.1. : $T^{-1}(B) \in \mathcal{B} \; \forall B \in \mathcal{B}$

$T^{-b}(B) \cap T^{-\ell}(B) = \emptyset$ if $b \neq \ell$. Indeed, if $y \in T^{-b}(B) \cap T^{-\ell}(B)$ (so $T^{\ell}(y) \in B$) for $\ell > b$, then $T^{b}(y) \in B$ and $T^{\ell-b}(T^{b}(y)) \in B$ which is a contradiction with the definition of B! Since $T_{*}\mu = \mu$, we have that $\mu(T^{-b}(B)) = \mu(B) \ \forall b \in \mathbb{N}$. Thus,

(μ fin. meas)

$$\infty > \mu(X) \geq \sum_{b=0}^{\infty} \mu(T^{-b}(B)) = \sum_{b=0}^{\infty} \mu(B).$$

($\mu(X) \geq \mu(\bigcup T^{-b}(B)) = \sum \mu(T^{-b}(B))$)

Hence $\mu(B) = 0$. The last claim (E). QED

4.1.2 Theorem Let X be a separable metric space, μ a finite Borel measure, $T: X \to X$ continuous and assume $T_{*}\mu = \mu$ then μ-a.e. points $x \in X$ are recurrent in the sense of Definition 2.1.1. In particular, $\operatorname{spt}\mu \subset \overline{R(T)}$.

proof: Since X is separable its topology has a countable basis $\{U_i\}_{i \in \mathbb{N}}$. Now x is recurrent $\Leftrightarrow$ for $U_i \ni x$ there exists $n \in \mathbb{N} \setminus \{0\}$ s.t. $T^{n}(x) \in U_i$. By Theorem 4.1.1, there exists a measurable set $\tilde{U}_i \subset U_i$ with $\mu(\tilde{U}_i) = \mu(U_i)$ and for all $y \in \tilde{U}_i$ there is $n \in \mathbb{N} \setminus \{0\}$ s.t. $T^{n}(y) \in U_i$. Now $x \in R(T) \Leftrightarrow x \in \tilde{U}_i$ for i s.t. $x \in U_i$. Thus

$$R(T) = \bigcap_{i=1}^{\infty} \tilde{U}_i \cup (X \setminus U_i).$$

Since $\mu(\tilde{U}_i \cup (X \setminus U_i)) = \mu(U_i \cup (X \setminus U_i)) = \mu(X)$

& $T^{n-k}(T^k(y)) \in B$ is contradict with def of B. Since $T_* \mu = \mu \Rightarrow \mu(T^{-k}(B)) = \mu(B)\ \forall k$

$$\Rightarrow \infty > \mu(X) \geq \sum_{k=0}^{\infty} \mu(T^{-k}(B)) = \sum \mu(B)$$

$$\Rightarrow \mu(B) = 0$$

X sep metric space, μ finite Borel measure, $T: X \to X$ & $T_* \mu = \mu \Rightarrow \mu$-a.e. $x \in X$ are recurrent in the sense of def 2.1.1

X is sep $\Rightarrow$ count. basis $\{U_i\}_{i \in \mathbb{N}}$.
x is recurrent if for U_i $\exists x$, $\exists n \in \mathbb{N} \setminus \{0\}$ s.t. $T^n(x) \in U_i$. By thm 4.1.1 we have $\mu(R(T)) = \mu(X)$. By the definition of spt, $\operatorname{spt}(\mu) \subset \overline{R(T)}$

QED

4.1.3 Definition Let T be a measure preserving transformation on a finite measure space $(X, \mathcal{B}, \mu)$ and assume $A \in \mathcal{B}$ satisfies $\mu(A) > 0$. Then A determines a derivative transformation $T_A : A \to A$ by

$$T_A(x) = T^n(x),$$

where $n \in \mathbb{N} \setminus \{0\}$ is the smallest integer s.t. $T^n(x) \in A$.

4.1.4 Remark The map T_A is called the first return map or the poincaré map. By Theorem 4.1.1 T_A is defined for μ-a.e. $x \in A$.

4.1.5 Definition Let $(X, \mathcal{B}, \mu, T)$ be a measurable dynamical system and let $f: X \to \mathbb{N}$ be measurable. Set

$$X_f := \{(x, b) \in X \times \mathbb{N} \mid x \in X, 1 \leq b < f(x)\}.$$

Define $\mathcal{B}_f = \mathcal{B} \times \mathcal{P}(\mathbb{N})$ where $\mathcal{P}(\mathbb{N})$ is the Borel σ-algebra on $\mathbb{N}$. Set $\mu_f := \mu \times \mathcal{H}^0$ ($\mathcal{H}^0$ counts el.ts in set). The primitive transformation of T is the map $T_f: X_f \to X_f$, where

$$T_f(x, b) = \begin{cases} (x, b+1) & \text{if } b < f(x) \\ (T(x), 1) & \text{if } b = f(x) \end{cases}$$

4.1.6 Remark Observe that $(T_f)_{X \times \{1\}} = T$. The maps T_A and T_f are called induced maps. Compare definitions 1.3.11 and 1.3.12.

if $\mu(X) < \infty$ and $f \in L^1(X, \mathcal{B}, \mu)$, then

$$\mu_f(X_f) = \int f(x)\, d\mu(x)$$

4.2 Ergodicity and mixing

4.2.1 Definition Let $(X, \mathcal{B}, \mu)$ be a measure space and $T: X \to X$ a measure preserving map. A measurable function $f: X \to \mathbb{R}$ is *essentially T-invariant* if

$$\mu(\{x \in X \mid f(T(x)) \neq f(x)\}) = 0$$

A set $A \in \mathcal{B}$ is *essentially T-invariant* if χ_A is essentially T-invariant. The map T is *ergodic* or μ is *ergodic* wrt T if $\mu(A) = 0$ or $\mu(A^c) = 0$ for all essentially T-invariant sets $A \in \mathcal{B}$.

4.2.2. Proposition Let $(X, \mathcal{B}, \mu, T)$ be a measurable dynamical system. Then

a) a set $A \in \mathcal{B}$ is essentially T-invariant $\iff \mu(T^{-1}(A) \Delta A) = 0$ where $A \Delta B := (A \setminus B) \cup (B \setminus A)$.

b) T is ergodic $\iff$ every essentially T-inv. $f: X \to \mathbb{R}$ is constant μ-a. everywhere.

lookb, ⇐ libe (b)

c) if $\mu(X) < \infty$, then T is ergodic $\Leftrightarrow$ every essentially T-invariant function $f \in L^p(X, \mu)$ is a constant μ-a. everywhere, where $1 \leq p \leq \infty$.

Proof: a) Since $T^{-1}(A) \Delta A = \{x \in X \mid \chi_A(T_A(x)) \neq \chi_A(x)\}$ the claim follows from the definition.

b) $\Leftarrow$ Let $A \in \mathcal{B}$ be essentially T-invariant. Then by definition, χ_A is essentially T-invariant and, thus, χ_A is a constant μ-a. everywhere (assumption). Therefore, $\mu(A) = 0$ or $\mu(A^c) = 0$. So T is ergodic.

$\overset{cp}{\Rightarrow}$ Let f be essentially T-invariant. If f is not a constant μ-a. everywhere, there is a $c \in \mathbb{R}$ s.t. ($\mathbb{R}$ in $\mathcal{B}$-algebra...)

$\mu(\{x \in X \mid f(x) > c\}) > 0$ and $\mu(\{x \in X \mid f(x) < c\}) > 0$.

Now

$$A = \{x \in X \mid f(x) > c\} \in \mathcal{B}$$

is essentially T-invariant since f is essentially T-invariant, but $\mu(A) > 0$ and $\mu(A^c) > 0$, which is a contradiction.

c) (E).

QED

4.2.3 Proposition Let $(X, \mathcal{B}, \mu, T)$ be a measurable

dynamical system and let $f: X \to \mathbb{R}$ be essentially T-inv. Then there exists a completely invariant measurable function $\tilde{f}: X \to \mathbb{R}$ s.t. $\tilde{f} = f$ μ-a. everywhere.

Proof: Let $A = \{x \in X \mid f(T(x)) = f(x)\}$. Then $A \in \mathcal{B}$, since, $A = (f \circ T - f)^{-1}(0)$. Further, $\mu(A^c) = 0$. Now $T^{-n}(A) \in \mathcal{B}$ for $n \in \mathbb{N}$ so

$$B := \{x \in X \mid f(T^n(x)) = f(x) \ \forall n \in \mathbb{N}\}$$

$$(E) \quad \bigcap_{n=0}^{\infty} T^{-n}(A) \in \mathcal{B}.$$

Moreover, $\mu(B^c) = 0$ since $T^{-n}(A)$ has full measure. Define $\tilde{f}: X \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathcal{B} \\ f(T^n(x)) & \text{if } x \in T^{-n}(B) \\ 0 & \text{otherwise} \end{cases}$$

Then $\tilde{f}$ is measurable and $\tilde{f} \circ T = \tilde{f}$ (E). QED

4.2.4 Definition Let $(X, \mathcal{B}, \mu, T)$ be a probability space and $T_* \mu = \mu$. The map T (or the measure μ) is (strong) mixing if

$$\lim_{n \to \infty} \mu(T^{-n}(A) \cap B) = \mu(A)\,\mu(B)$$

$\forall A, B \in \mathcal{B}$. The map T (or the measure μ) is weak mixing

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} |\mu(T^{-1}(A)\cap B) - \mu(A)\mu(B)| = 0$$

$\forall A, B \in \mathcal{B}$

4.2 Theorem Let $(X, \mathcal{B}, \mu, T)$ be a probability space (i.e. $\mu(X)=1$) with T measure preserving, $T_*\mu = \mu$. The map T is mixing / weak mixing

$\Leftrightarrow$

$$\lim_{n\to\infty} \int f(T^n(x))\, g(x)\, d\mu(x) = \int f(x)\, d\mu(x) \int g(x)\, d\mu(x) /$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \left| \int f(T^n(x))\, g(x)\, d\mu(x) - \int f(x)\, d\mu(x) \int g(x)\, d\mu(x) \right| = 0$$

for all bounded measurable functions f and g. If T is mixing, it is weak mixing. If T is weak mixing it is ergodic.

proof: $\Leftarrow$ Choose $f = \chi_A$ and $g = \chi_B$.

$\Rightarrow$ By assumption, the claim is true for characteristic functions of sets in the σ-algebra $\mathcal{B}$, so it is true for finite linear combinations of these, that is, for simple functions. Assume that $0 \le f, g \le M$. Let $\varepsilon > 0$ and let $f_\varepsilon \le f$ and $g_\varepsilon \le g$ be simple functions s.t.

$\int |f - f_\varepsilon|\, d\mu < \varepsilon$ and $\int |g - g_\varepsilon|\, d\mu < \varepsilon$.

↳ just measure theory

For the mixing part:

$$\lim_{n\to\infty}\int f(T^n(x))\,g(x)\,d\mu(x) \geq \lim_{n\to\infty}\int f_\beta(T^n(x))\,g_\beta(x)\,d\mu(x)$$

f_β, g_β simple

$$= \int f_\beta(x)\,d\mu(x)\int g_\beta(x)\,d\mu(x)$$

$$\geq \left(\int f\,d\mu-\varepsilon\right)\left(\int g\,d\mu-\varepsilon\right)$$

$$= \int f\,d\mu\int g\,d\mu-\varepsilon\left(\int f\,d\mu+\int g\,d\mu\right)+\varepsilon^2.$$

Similarly,

$f = f-f_\beta+f_\beta,\ g = g-g_\beta+g_\beta$

$$\lim_{n\to\infty}\int f(T^n(x))\,g(x)\,d\mu(x) \leq \lim_{n\to\infty}\int f_\beta(T^n(x))\,g_\beta(x)\,d\mu(x)$$

$$+M\int |f(T^n(x))-f_\beta(T^n(x))|\,d\mu(x)$$

$$+M\int |g(x)-g_\beta(x)|\,d\mu(x)$$

$T_*\mu=\mu \Leftrightarrow \int f\,d(T_*\mu)=\int f\circ T\,d\mu$ (measure theory)

$$\leq \int f_\beta\,d\mu\int g_\beta\,d\mu+2M\varepsilon$$

$$\leq \int f\,d\mu\int g\,d\mu+2M\varepsilon.$$

Letting $\varepsilon\to 0$ gives the claim.

$$\frac{1}{n}\sum_{j=0}^{n-1} = \frac{1}{n}\sum_{j=0}^{N} + \frac{1}{n}\sum_{j=N}^{n-1}$$ Sonmixing $\Rightarrow$ weak mixing.

Weak mixing $\Rightarrow$ ergodic, let $A\in\mathcal{F}$ be essentially T-invariant, that is, $\mu(T^{-1}(A)\,\Delta\,A)=0$. Set $B=A$. Then by weak mixing

$$0=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\left|\mu(T^{-1}(A)\cap A)-\mu(A)^2\right|$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\left|\mu(A)-\mu(A)^2\right|$$

$$=\lim_{n\to\infty}\left|\mu(A)-\mu(A)^2\right| \overset{\text{prob measure}}{=} \lim_{n\to\infty}\left(\mu(A)-\mu(A)^2\right).$$

Thus, $\mu(A)=\mu(A)^2$, that is, $\mu(A)=0$ or $\mu(A)=1$. So μ is ergodic

QED ∎

4.2.6 Theorem Let X be a compact metric space, $T: X \to X$ be a continuous map and μ be a T-invariant Borel probability measure. Then

a) if T is ergodic, then $\overline{\mathcal{O}^+(x)} \cap \operatorname{spt}\mu = \operatorname{spt}\mu$ for μ-a.e. $x \in X$.

b) if T is mixing, then T is topologically mixing on $\operatorname{spt}\mu$.

Proof: a) Let U be open and set

$$B = B(U) := \bigcup_{s=0}^{\infty} T^{-s}(U).$$

Now $T^{-1}(B) \Delta B = U \setminus \bigcup_{s=1}^{\infty} T^{-s}(U)$. By Theorem 4.1.1, we have that $\mu(T^{-1}(B) \Delta B) = 0$. By proposition 4.2.2, B is essentially invariant. Since T is ergodic, $\mu(B) \in \{0, 1\}$. Let $\{U_i\}_{i \in \mathbb{N}}$ be a basis of the topology of X. If $U_i \cap \operatorname{spt}\mu \neq \emptyset$ then $\mu(U_i) > 0$, giving $\mu(B(U_i)) = 1$. Therefore, $\mathcal{O}^+(x) \cap U_i \neq \emptyset$ for μ-a.e. $x \in X$. This implies that for μ-a.e. $x \in X$ $\mathcal{O}^+(x) \cap U_i \neq \emptyset$ $\forall i \in \mathbb{N}$ s.t. $U_i \cap \operatorname{spt}\mu \neq \emptyset$. Thus $\overline{\mathcal{O}^+(x)} \cap \operatorname{spt}\mu = \operatorname{spt}\mu$ for μ-a.e. $x \in X$

b) (E)

QED ∎

example of ergodic but not mixing

4.2.7. Example The rotation $R_\alpha : S^1 \to S^1$ is ergodic

w.r.t. $\vec{\ell}|_{S^1} =: \ell \iff \alpha \notin \mathbb{Q}$.

proof: $\Leftarrow$ By Prop 4.2.2. (<), it is enough to show that every R_α-invariants function $f \in L^2(S^1, \mathbb{C})$ is constant. Write (magic) $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2n\pi x i}$ as a Fourier series (converges in L^2)

Do you condition bcs $f \in L^2$

Now

$$f \circ R_\alpha(x) = \sum_{n=-\infty}^{\infty} a_n e^{2n\pi(x+\alpha)i}.$$

you don't need mod 1

Since $f \circ R_\alpha = f$ (f is R_α-inv), we have that $a_n = a_n e^{2n\pi\alpha i}$ $\forall n \in \mathbb{Z}$, by the uniqueness of the Fourier coefficients. Since $\alpha \notin \mathbb{Q}$, we have that $e^{2n\pi\alpha i} \neq 1$ for all $n \neq 0$, so $a_n = 0$ $\forall n \neq 0$.

Thus $f = a_0$ is constant.

$\Rightarrow$ (E)

QED ∎

4.2.8 Example The map $E_m: [0,1]/_{\sim 1} \to [0,1]/_{\sim 1}$ is mixing w.r.t. ℓ.

proof: Since the collection

$$\mathcal{C} := \{[p\,m^{-i}, (p+1)\,m^{-i}] \mid p \in \mathbb{Z}, i \in \mathbb{N}\}$$

is sufficient, it is enough to study intervals

$A = [p\,m^{-i}, (p+1)\,m^{-i}]$ and $B = [q\,m^{-j}, (q+1)\,m^{-j}]$.

See exercises next weeks. Since $E_m^{n}(A)$ consists of

m evenly distributed intervals of length $m^{-(i+1)}$, $E_m^{-n}(A)$ consits of m^n evenly distributed intervals of length $m^{-(i+n)}$, so when $n > j$, the distance between two successive intervals is $m^{-n} < m^{-j}$, so $E_m^{-n}(A) \cap B$ consits of m^{n-j} intervals of length $m^{-(i+n)}$. Thus

$$\mathcal{L}(E_m^{-1}(A) \cap B) = m^{n-j} m^{-(i+n)} = m^{-i-j} = \mathcal{L}(A)\mathcal{L}(B).$$

4.2.9 Example Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{Z}$, $|\det A| = 1$ (no 1 eigv < 1 and one eigv > 1) and $|\sigma| \neq 1$, for both eigenvalues σ of A. Then $A : T^2 \to T^2$ is ergodic w.r.t. $\mathcal{L}^2$.

Via 4.2.4 (c)

Proof: Let $f \in L^2(T^2, \mathcal{L}^2)$ be invariant. Representing f and $f \circ A$ as Fourier series, we obtain

$$f(x,y) = \sum_{m,n=-\infty}^{+\infty} a_{m,n} e^{2\pi i (mx+ny)},$$

$$f \circ A(x,y) = \sum_{m,n=-\infty}^{+\infty} a_{m,n} e^{2\pi i ((ma+nc)x + (mb+nd)y)}$$

Since $f \overset{\text{inv.}}{=} f \circ A$, the uniqueness of the Fourier coefficients implies

$$a_{m,n} = a_{ma+nc,\, mb+nd}, \text{ that is}$$

$a_{m,n} =: a_{\binom{m}{n}} = a_{A^T\binom{m}{n}}$

Since $|\sigma| \neq 1$, for all $(m,n) \neq (0,0)$ either

$$(A^T)^b \binom{m}{n} \xrightarrow{b \to \infty} \infty \quad \text{or}$$

$$(A^T)^{-b} \binom{m}{n} \xrightarrow{b \to \infty} \infty$$

If $a_{m,n} \neq 0$, there is a sequence $(m_i, n_i) \to \infty$ with $a_{m_i,n_i} = a_{m,n}$. In particular $|a_{0,0}| \not\to 0$ $(0,0)\to\infty$, which is a contradiction since $f \in L^2(T^2, \mathbb{C}^2)$. So $a_{m,n} = 0$ for all $(m,n) \neq (0,0)$. That is, $f = a_{0,0}$ is a constant. So the claim follows from Proposition 4.2.2(c).

QED

Markov Measures

Let A be an $m \times m$ stochastic matrix. Assume that A has

↦ so not $A(\ ) = (\ )$, but $(\ )A = (\ )$

a nonnegative left eigenvector with eigenvalue 1, that is, $wA = w$. Equivalently, $A^T w^T = w^T$. Normalise w s.t. $\sum_{i=1}^{m} w_i = 1$. The pair (A, w) is called a Markov chain. We define a Borel probability measure $P_{A,w}$ on Σ_m (Σ_m^+) as follows: for all $n \in \mathbb{Z}$ (or $\mathbb{N}$), set $P_{A,w}(C_{j_n}) = w_{j_n}$

↦ n'th place a j

where $C_{j_n} = \{x \in \Sigma_m \mid x_n = j_n\}$. For all $b \in \mathbb{N}$, set $P_{A,w}(C_{j_n \cdots j_{n+b}}) = w_{j_n} \prod_{i=n}^{n+b-1} A_{j_i j_{i+1}}$. We can interpret this as follows: w gives the initial probality distribution on $\{0,1,\dots,m-1\}$ and A_{ij} gives the transition probability from i to j. $P_{A,w}(C_{j_n})$ is the probability of observing the letter j_n as the n^{th} coordinate of randomly chosen sequence. Since $wA = w$

we have that

$$P_{A,w}(C_{j_{n+1}}) = w_{j_{n+1}} = (wA)_{j_{n+1}} \overset{\text{def product vector matrix}}{=} \sum_{i_n=0}^{m-1} w_{i_n} A_{i_n j_{n+1}} = \sum_{i_n=0}^{m-1} P_{A,w}(C_{i_n}) A_{i_n j_{n+1}}$$

$$= \sum_{i_n=0}^{m-1} P_{A,w}(C_{i_n j_{n+1}}).$$

The measure $P_{A,w}$ extends naturally to the Borel σ-algebra on Σ_m and it is called a Markov measure. Clearly, $\sigma_* P_{A,w} = P_{A,w}$.

An important special case is when A_{ij} does not depend on i, that is

$$A = \begin{pmatrix} p_0 & p_1 & \cdots & p_{m-1} \\ \vdots & \vdots & & \\ p_0 & p_1 & \cdots & p_{m-1} \end{pmatrix}.$$

Then $P_{A,w}$ is the product measure $P_{A,w} = \prod_{\mathbb{Z}} \nu$, where $\nu(i) = p_i$ for $i = 0, \dots, m-1$. In this case, $P_{A,w}$ is called a Bernoulli measure.

4.2.10 Theorem Let A be an irreducible stochastic matrix and let

$$\tilde{A}_{ij} = \begin{cases} 0 & \text{if } A_{ij} = 0 \\ 1 & \text{if } A_{ij} > 0 \end{cases}$$

be an adjacency matrix. Then $\operatorname{spt}(P_{A,w}) = \Sigma_{\tilde{A}}^{\nu} \subset \Sigma_m$. If A is primitive, then σ is mixing w.r.t. $P_{A,w}$.

proof: The first claim is an exercise.

Assume that A is primitive. Then w is unique and $w_i > 0$ for all $i \in \{0, \dots, m-1\}$ by corollary 3.3.5. By exercise 4.1.11 it is enough to prove the mixing property for the sufficient collection of cylinders of the form $C_{\alpha_{-\ell} \dots \alpha_\ell}$ and $C_{\beta_{-b} \dots \beta_\ell}$. Let $n > b + \ell + 1$. Then

$$\sigma^{-n}(C_{\beta_{-b} \dots \beta_b}) \cap C_{\alpha_{-\ell} \dots \alpha_\ell} = \bigcup_{\substack{\gamma_i \in \{0,\dots,m-1\} \\ i = \ell+1, \dots, n-b-1}} C_{\alpha_{-\ell} \dots \alpha_\ell \gamma_{\ell+1} \dots \gamma_{n-b-1} \beta_{-b} \dots \beta_b}$$

Now by construction,

$$P_{A,w}(C_{\alpha_{-\ell} \dots \beta_b}) = \underbrace{w_{\alpha_{-\ell}}}_{\text{by def of } P_{A,w}} \prod_{i=-\ell}^{\ell-1} A_{\alpha_i \alpha_{i+1}} A_{\alpha_\ell \gamma_{\ell+1}} \times$$

$$\prod_{i=\ell+1}^{n-b-\ell-1} A_{\gamma_i \gamma_{i+1}} \cdot A_{\gamma_{n-b-\ell} \beta_{-b}} \prod_{i=-b}^{b} A_{\beta_i \beta_{i+1}}$$

$$= P_{A,w}(C_{\alpha_{-\ell} \dots \alpha_\ell}) A_{\alpha_\ell \gamma_{\ell+1}} \prod_{i=\ell+1}^{n-b-\ell-1} A_{\gamma_i \gamma_{i+1}} A_{\gamma_{n-b-\ell} \beta_{-b}} \underbrace{w_{\beta_{-b}}^{-1}}_{\text{b/c } w_i > 0 \text{ so ok!}} P_{A,w}(C_{\beta_{-b} \dots \beta_b})$$

Summing over γ's and rearranging

$$P_{A,w}(\sigma^{-n}(C_{\beta_{-b} \dots \beta_b}) \cap C_{\alpha_{-\ell} \dots \alpha_\ell})$$

$$= P_{A,w}(C_{\alpha_{-\ell} \dots \alpha_\ell}) P_{A,w}(C_{\beta_{-b} \dots \beta_b}) w_{\beta_{-b}}^{-1} A^{n-b-\ell}_{\alpha_\ell \beta_{-b}}.$$

So it is enough to show that

$$\lim_{n \to \infty} w_{\beta_{-b}}^{-1} (A^{n-b-\ell})_{\alpha_\ell \beta_{-b}} = 1.$$

As in the proof of Theorem 3.5.10, we see

$$\frac{v A^n}{d^n} \xrightarrow[n \to \infty]{} \alpha w \text{ for all nonnegative } v.$$

since A is stochastic, $d = 1$ and, choosing $v = e_j$, we have

$A^n_{ji} \xrightarrow[n \to \infty]{} \alpha w_i$, $\forall i, j \in \{0, \dots, m-1\}$. Therefore

$\lim_{n\to\infty} A^{n-b-\ell}_{\alpha_\ell \beta_{-b}} = \alpha\, w_{\beta_{-b}}$. Since A is stochastic, we have $\sum_{i=0}^{m-1} A^n_{j_i} = 1 \quad \forall n \in \mathbb{N}$ and $j \in \{0,\dots,m-1\}$. Thus, $\sum_{i=0}^{m-1} \alpha\, w_i = 1$, which implies that $\alpha = 1$. So $\lim_{n\to\infty} A^{n-b-\ell}_{\alpha_\ell \beta_{-b}}\, w^{-1}_{\beta_{-b}} = \alpha\, w_{\beta_{-b}} w_{\beta_{-b}}^{-1} = 1$

QED

4.3 Ergodic theorems

4.3.1 Theorem (Von Neumann ergodic theorem) Let U be a linear isometric on a separable Hilbertspace H and let P be the orthogonal projection onto the space $I = \{v \in H \mid Uv = v\}$ of U-invariant vectors. Then $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} U^i v = Pv \quad \forall v \in H$. (average over i)

Proof: omitted.

4.3.2 Corollary Let $(X, \mathcal{B}, \mu)$ be a finite measure space and let $T: X \to X$ be a measurable map s.t. $T_* \mu = \mu$. Let $f \in L^2(X, \mu)$. Then

$$\lim_{n\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x)) \xrightarrow{L^2} \bar{f} \in L^2(X, \mu)$$

Proof: (E)

QED

4.3.3 Definition Let $a_1, \dots, a_m \in \mathbb{R}$ and $1 \le n \le m$.

The number a_s is an n-leader if

$$a_s + a_{s+1} + \dots + a_{s+p-1} \geq 0$$

for some $1 \leq p \leq n$.

4.3.4 Lemma For all $1 \leq n \leq m$, the sum of n-leaders is nonnegative.

Proof: (E) QED

4.3.5 Theorem (Maximal ergodic theorem) Let $(X, \mathcal{B}, \mu)$ be a finite measure space, $T: X \to X$ a measure preserving map and $f \in L^1(X, \mu)$. Then

$$\int_A f(x)\, d\mu(x) \geq 0,$$

where $A = \{x \in X \mid \sum_{i=0}^{b} f(T^i(x)) \geq 0 \text{ for some } b \in \mathbb{N}\}$

Proof: Let

$$A_n = \{x \in X \mid \sum_{i=0}^{b} f(T^i(x)) \geq 0 \text{ for some } 0 \leq b \leq n\}.$$

It is clear that $A_n \subset A_{n+1}$, and $A = \bigcup_{n=0}^{\infty} A_n$. So it is enough to prove that $\int_{A_n} f(x)\, d\mu(x) \geq 0 \quad \forall n \in \mathbb{N}$, since $f\chi_{A_n} \to f\chi_A$ and $|f\chi_{A_n}| \leq |f\chi_A|$ (Dominated convergence theorem). Fix $m \in \mathbb{N}$. Let $s_n(x)$ be the sum of n-leaders

of the sequence

$$f(x),\ f(T(x)),\ \dots,\ f(T^{m+n-1}(x)).$$

For all $b \le m+n-1$, let

$$B_b = \{x \in X \mid f(T^b(x)) \text{ is an } n\text{-leader}\}.$$

By Lemma 4.3.4, we have that

$$0 \le \int_X s_n(x)\, d\mu(x) = \sum_{b=0}^{n+m-1} \int_{B_b} f(T^b(x))\, d\mu(x) \quad (*)$$

Since $x \in B_b \iff T(x) \in B_{b-1}$, then $B_b = T^{-b}(B_{b-1})$ and further, $B_b = T^{-b}(B_0)$ $\forall\, 1 \le b \le m-1$. Thus,

$$\int_{B_b} f(T^b(x))\, d\mu(x) = \int_{T^{-b}(B_0)} f(T^b(x))\, d\mu(x)$$

$$= \int f \chi_{B_0} \circ T^b \, d\mu = \int f \chi_{B_0}\, d\mu = \int_{B_0} f\, d\mu.$$

So the first n terms in $(*)$ are equal. Since $B_0 = A_n$, we have

$$(**)\quad m \int_{A_n} f\, d\mu + n \int |f|\, d\mu \ge 0,$$

if $\int_{A_n} f\, d\mu < 0$, choose m big enough s.t. $(**) < 0$ ↯

so $\int_{A_n} f\, d\mu \ge 0$

QED ∎

4.3.6 Theorem (Birkhoff's ergodic theorem) Let (X, β, μ) be a finite measure space, $T: X \to X$ a measure preserving map and $f \in L^1(X, \mu)$. Then the limit

if ergodic $\Rightarrow$ $\bar{f}$ is const

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) =: \lim_{n\to\infty} f_n(x) = \bar{f}(x) \quad (*)$$

exists for μ-a.e. $x \in X$, $\bar{f} \in L^1(X,\mu)$, $\bar{f} \circ T(x) = \bar{f}(x)$

for μ-a.e. $x \in X$, $f_n \xrightarrow{L^1} \bar{f}$ and $\int_A \bar{f}\, d\mu = \int_A f\, d\mu$

L^1 convergence only pointwise

for all essentially invariant sets $A \in \mathcal{B}$.

Proof: First we prove that the limit exists. Let $a, b \in \mathbb{R}$ with $a < b$, set

(p want measure 0 (points where limit does not exist))

$$X(a,b) = \{x \in X \mid \liminf f_n(x) < a < b < \limsup f_n(x)\}.$$

Then $X(a,b) \in \mathcal{B}$ and $T(X(a,b)) = X(a,b)$. Applying Theorem 4.3.5 to $T|_{X(a,b)}$ and $f - b$, we obtain

$$\int_{X(a,b)} f(x) - b \, d\mu(x) \geq 0.$$

Similarly, we have that

$$\int_{X(a,b)} a - f(x) \, d\mu(x) \geq 0.$$

So

$$\int_{X(a,b)} a - b \, d\mu(x) \geq 0.$$

Therefore, $\mu(X(a,b)) \overset{a<b}{=} 0$. Since $\{x \in X \mid (*) \text{ is not valid}\}$

$= \bigcup_{\substack{a,b\in\mathbb{R} \\ a<b}} X(a,b)$, we have proved $(*)$. By Fatous lemma,

$$\int |\bar{f}|\, d\mu = \int \liminf_{n\to\infty} |f_n|\, d\mu \overset{\text{Fat.}}{\leq} \liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int |f(T^k(x))|\, d\mu(x)$$

$$= \liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int |f|\, d\mu = \int |f|\, d\mu.$$

Thus $\bar{f} \in L^1(X,\mu)$. The claim $\bar{f} \circ T = \bar{f}$ is true at

all points where the limit exists.